HOCHSCHILD LEFSCHETZ CLASS FOR $\mathcal{D}$-MODULES

AJAY RAMADOSS, XIANG TANG, AND HSIAN-HUA TSENG

ABSTRACT. We introduce a notion of Hochschild Lefschetz class for a good coherent $\mathcal{D}$-module on a compact complex manifold, and prove that this class is compatible with the direct image functor. We prove an orbifold Riemann-Roch formula for a $\mathcal{D}$-module on a compact complex orbifold.

1. Introduction

In [KS], Kashiwara and Schapira systematically studied the Hochschild class for deformation quantization algebroids. As an application, they obtained a new way to define the Euler class of a good coherent $\mathcal{D}$-module on a complex manifold, introduced by Schapira and Schneiders [SS2]. In this paper, we aim to generalize the notion of Hochschild class to the equivariant setting.

Let M be a compact complex manifold and $\mathcal{D}_M$ the sheaf of holomorphic differential operators on M. A coherent $\mathcal{D}_M$-module $\mathcal{M}$ is called "good" if for any compact subset of M there is a neighborhood in which $\mathcal{M}$ admits a finite filtration $(\mathcal{M}_k)$ by coherent $\mathcal{D}_M$-submodules such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration. We denote by $D^b(\mathcal{D}_M)$ the bounded derived category of $\mathcal{D}_M$-modules, and $D^b_{\mathrm{coh}}(\mathcal{D}_M)$ the full triangulated subcategory of $D^b(\mathcal{D}_M)$ consisting of objects with coherent cohomologies. Let $X := T^*M$ be the cotangent bundle of M. Following [KS], we consider the sheaf $\widehat{\mathcal{E}}_X$ of formal microdifferential operators on X. Let $\pi_M : X = T^*M \to M$ be the canonical projection. There is a natural flat embedding map $\pi_M^{-1}\mathcal{D}_M \hookrightarrow \widehat{\mathcal{E}}_X$. This gives a natural functor from $D^b_{\mathrm{coh}}(\mathcal{D}_M)$ to $D^b_{\mathrm{coh}}(\widehat{\mathcal{E}}_X)$. Such a functor allows the use of microlocal techniques to study $\mathcal{D}_M$-modules. Let $\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X)$ be the $\mathbb{C}$-sheaf[1] of Hochschild homologies of $\widehat{\mathcal{E}}_X$ on X. For $\mathcal{M} \in D^b_{\mathrm{coh}}(\mathcal{D}_M)$ and an element $u \in Hom_{\mathcal{D}_M}(\mathcal{M}, \mathcal{M})$, Kashiwara and Schapira [KS] introduced a *Hochschild class*

$$hh(\mathcal{M}, u) \in H^0_{\mathrm{supp}(\mathcal{M})}(X; \mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X)).$$

The image of the the Hochschild class $hh(\mathcal{M}, u)$ under the quasi-isomorphism

$$\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X) \to \mathbb{C}[\dim_X]$$

is called the *Euler class* of $(\mathcal{M}, u)$. As Hochschild homology behaves well under direct images, the Hochschild class $hh(\mathcal{M}, u)$ satisfies a nice formula [KS, Theorem 4.3.5] under the direct image functor. This formula is analogous to the direct image property of the Euler class of $(\mathcal{M}, u)$, which was proved by Schapira and Schneiders [SS2].

In this paper we consider a holomorphic diffeomorphism γ on M. Let $\mathcal{M}$ be a $\mathcal{D}_M$-module. Then, the sheaf $\gamma_*\mathcal{M}$ of $\mathbb{C}$-vector spaces on M has a natural $\mathcal{D}_M$-module structure. This in turn gives a natural functor $\gamma_* : D^b_{\mathrm{coh}}(\mathcal{D}_M) \to D^b_{\mathrm{coh}}(\mathcal{D}_M)$. A similar construction and functor can be introduced for $\widehat{\mathcal{E}}_X$-modules. Given an element $u \in Hom_{\mathcal{D}_M}(\mathcal{M}, \gamma_*(\mathcal{M}))$, we introduce a *Hochschild Lefschetz class*[2]

$$hh^\gamma(\mathcal{M}, u) \in H^0(X, \mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X))$$

Key words and phrases. equivariant Hochschild class, D-module.

[1] This is actually a slight abuse of terminology. In fact, $\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X)$ is an object in the derived category of sheaves of $\mathbb{C}$-vector spaces on X.

[2] See Eq. (1) for the definition of $\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X)$.

for $X = T^*M$. Our construction coincides with the Kashiwara-Schapira Hochschild class $hh(\mathcal{M}, u)$ when $\gamma = id$. The *Lefschetz class* of u is defined to be the image of $hh^\gamma(\mathcal{M}, u)$ under the quasi-isomorphism

$$\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X^\gamma) \to \iota_!(\mathbb{C}_{X^\gamma}[\dim(X^\gamma)]).$$

Like the Hochschild class, we prove that the Hochschild Lefschetz class satisfies nice formulas under the direct image functor. We expect that this approach will provide a relatively easy route to results about the Lefschetz class introduced by Guillermou [G]. Let Γ be a finite group acting on M by holomorphic diffeomorphisms. We apply our developments to study the Hochschild class and the Euler class of a good Γ-equivariant coherent $\mathcal{D}_M$-module $\mathcal{M}$. In this situation, every $\gamma \in \Gamma$ naturally defines an element γ in $Hom_{\mathcal{D}_M}(\mathcal{M}, \gamma_*(\mathcal{M}))$. We can use the expression

$$\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} hh^\gamma(\mathcal{M}, \gamma)$$

to introduce the orbifold Hochschild class of $\mathcal{M}$ on the quotient orbifold $Q_X := X/\Gamma = T^*M/\Gamma$. We prove that this description of the orbifold Hochschild class for $\mathcal{M}$ is equivalent to the more abstract definition that arises by working with the sheaf of algebras $\mathcal{D}_M \rtimes \Gamma$ (and $\widehat{\mathcal{E}}_X \rtimes \Gamma$) over $Q_M := M/\Gamma$ (and Q_X) using techniques developed by Bressler, Nest, and Tsygan [BNT].

The main result of this paper is a Riemann-Roch formula for the Euler class of a good Γ-equivariant coherent $\mathcal{D}_M$-module $\mathcal{M}$ on M. We prove that (see Theorem 4.2)

$$\mathrm{eu}_Q(\mathcal{M}) = \Big(\frac{1}{m}\mathrm{ch}_Q(\sigma_{\mathrm{char}(\mathcal{M})}(\mathcal{M})) \wedge \mathrm{eu}_Q(N) \wedge \pi_M^* Td(IQ_M)\Big)_{\dim(IQ_X)}.$$

Hereby, IQ_X (and IQ_M) is the inertia orbifold associated to the orbifold Q_X (and Q_M); ch_Q is the orbifold Chern character for the orbifold K-theory element $\sigma_{ch(\mathcal{M})}(\mathcal{M})$; $\mathrm{eu}_Q(N)$ is a characteristic class associated to the normal bundle N of the local embedding $IQ \to Q$; $Td(IQ_M)$ is the Todd class of the orbifold bundle TIQ_M over IQ_M; π_M is the canonical projection from IQ_X to IQ_M; and m is the locally constant function on IQ_X measuring the size of the isotropy group at each point. The proof generalizes the original idea of Bressler, Nest, and Tsygan [BNT] along the developments in [PPT1] and [PPT2].

The paper is organized as follows. We start with fixing some basic notations in Sec. 2. In Sec. 3, we introduce the construction of the Hochschild Lefschetz class and orbifold Euler class for a good coherent $\mathcal{D}_M$-module, and discuss their properties. In Sec. 4, we explain the computation of the orbifold Euler (Chern) class.

Acknowledgments: We would like to thank Pierre Schapira and Giovanni Felder for interesting discussions. A.R. is supported by the Swiss National Science Foundation for the project "Topological quantum mechanics and index theorems" (Ambizione Beitrag Nr. PZ00P2_127427/1). X.T. is partially supported by NSF grant DMS-0900985.

2. Basic Notations

Throughout this paper, we closely follow the terminologies and conventions introduced in [KS]. Let M be a complex manifold of complex dimension d_M. In this paper, dimension of a complex manifold/orbifold always refers to its complex dimension. Consider $\mathcal{D}_M$ the sheaf of holomorphic differential operators on M and $\mathcal{D}_{M^a}$ the sheaf of holomorphic differential operators on M with the opposite algebra structure from $\mathcal{D}_M$. A coherent $\mathcal{D}_M$-module $\mathcal{M}$ is called "good" if in a neighborhood of any compact subset of M, $\mathcal{M}$ admits a finite filtration $(\mathcal{M}_k)$ by coherent $\mathcal{D}_M$-submodules such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ can be endowed with a good filtration (see [K, Definition 4.24]). We denote by $D^b(\mathcal{D}_M)$ the bounded derived category of $\mathcal{D}_M$ modules, and $D^b_{\mathrm{coh}}(\mathcal{D}_M)$ the full triangulated subcategory of $D^b(\mathcal{D}_M)$ consisting of objects with coherent cohomologies.

Let $X = T^*M$ be the cotangent bundle with dimension $d_X = 2d_M$ with the projection $\pi_M : X = T^*M \to M$. Denote by Ω^i_X the sheaf of holomorphic i-forms on X. On $X = T^*M$, consider the filtered sheaf $\widehat{\mathcal{E}}_X$ of $\mathbb{C}$-algebras of formal microdifferential operators, and the subsheaf $\widehat{\mathcal{E}}(0)_X$ of operators of order ≤ 0. Let $\pi_M^{-1}\mathcal{D}_M$ be the pullback of $\mathcal{D}_M$ on X. Denote by $D^b(\widehat{\mathcal{E}}_X)$ (and $D^b_{\mathrm{coh}}(\widehat{\mathcal{E}}_X)$) the bound derived category of $\widehat{\mathcal{E}}_X$-modules (and the subcategory of objects with coherent cohomologies.) There is a natural morphism $\pi_M^{-1}\mathcal{D}_M \hookrightarrow \widehat{\mathcal{E}}_X$ such that $\widehat{\mathcal{E}}_X$ is flat over $\pi_M^{-1}\mathcal{D}_M$. Given a coherent $\mathcal{D}_M$-module $\mathcal{M}$,

$$\widehat{\mathcal{M}} := \widehat{\mathcal{E}}_X \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{M}$$

defines a coherent $\widehat{\mathcal{E}}_X$-module. In this paper, we will mainly work with the $\widehat{\mathcal{E}}_X$-module $\widehat{\mathcal{M}}$ associated to $\mathcal{M}$. The support of $\widehat{\mathcal{M}}$ is called the *characteristic variety* of $\mathcal{M}$ and denoted by $\mathrm{char}(\mathcal{M})$. Denote by $\widehat{\mathcal{E}}_{X^a}$ the sheaf of formal microdifferential operators on $X = T^*M$ with the opposite algebra structure from $\widehat{\mathcal{E}}_X$.

Define $\omega := \Omega_X^{d_X}[d_X]$, and define the duality functor $D'_{\widehat{\mathcal{E}}_X}$ by

$$D'_{\widehat{\mathcal{E}}_X}(\mathcal{M}) := R\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\mathcal{M}, \widehat{\mathcal{E}}_X) \in D^b(\widehat{\mathcal{E}}_{X^a}), \quad \text{for an } \widehat{\mathcal{E}}_X\text{-module } \mathcal{M}.$$

Let Γ be a finite group acting on M holomorphically and also on $X = T^*M$. Note that for any $\gamma \in \Gamma$, one has a natural isomorphism $\gamma^{-1}\widehat{\mathcal{E}}_X \to \widehat{\mathcal{E}}_X$ of sheaves of $\mathbb{C}$-algebras on X. Hence, for any $\gamma \in \Gamma$, any $\widehat{\mathcal{E}}_X$-module $\mathcal{M}$ has the natural structure of a $\gamma^{-1}\widehat{\mathcal{E}}_X$-module. Consequently, the pushforward $\gamma_*\mathcal{M}$ (in the category of sheaves of $\mathbb{C}$-vector spaces on X) has the natural structure of a $\widehat{\mathcal{E}}_X$-module. It is easy to verify that the (right derived functor of) γ_* gives rise to a functor γ_* from $D^b(\widehat{\mathcal{E}}_X)$ (resp., $D^b_{\mathrm{coh}}(\widehat{\mathcal{E}}_X)$) to itself.

We denote by $\delta : X \to X \times X$ the diagonal embedding. For $\gamma \in \Gamma$, let

$$\delta_X^\gamma : X \to X \times X,\ \delta_X^\gamma(x) := (\gamma(x), x)$$

be the graph of the action of γ. Let $\mathcal{C}_X$ be the $\widehat{\mathcal{E}}_{X\times X^a}$-module $\delta_{X,*}\widehat{\mathcal{E}}_X$, and let $\mathcal{C}_X^\gamma$ be the $\widehat{\mathcal{E}}_{X\times X^a}$-module $\delta^\gamma_{X,*}\widehat{\mathcal{E}}_X$. The sheaf of Hochschild homologies[3] $\mathcal{HH}(\widehat{\mathcal{E}}_X)$ (resp., γ-twisted Hochschild homologies $\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X^\gamma)$) is defined to be the object

$$(1)\quad \mathcal{HH}_X(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X) := \delta_X^{-1}(\mathcal{C}_{X^a} \overset{L}{\otimes}_{\widehat{\mathcal{E}}_{X\times X^a}} \mathcal{C}_X) \quad \left(\text{resp.,}\quad \mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X^\gamma) := {\delta_X}^{-1}(\mathcal{C}_{X^a} \overset{L}{\otimes}_{\widehat{\mathcal{E}}_{X\times X^a}} \mathcal{C}_X^\gamma)\ \right)$$

of the derived category of sheaves of $\mathbb{C}$-vector spaces on X. For any object $\mathcal{F}$ in the derived category of sheaves of $\mathbb{C}$-vector spaces on X, $H^\bullet(X;\mathcal{F})$ shall denote the *hypercohomology* of X with coefficients in $\mathcal{F}$.

The completed tensor products $\underline{\otimes}$, $\underline{\boxtimes}$, etc. have exactly the same meaning as in [KS].

3. Lefschetz class

3.1. **Definition of Lefschetz class.** Let $\mathcal{M}$ be a good coherent $\mathcal{D}_M$ module on M. We apply the functor

$$\mathcal{M} \mapsto \widehat{\mathcal{M}} := \widehat{\mathcal{E}}_X \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{M}$$

and work with the corresponding $\widehat{\mathcal{E}}_X$-module $\widehat{\mathcal{M}}$. Let γ act on M holomorphically. Lift the action of γ to an action on $X := T^*M$. An element u in $Hom_{\mathcal{D}_M}(\mathcal{M}, \gamma_*(\mathcal{M}))$ naturally defines an element $\hat{u}$ in $Hom_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}, \gamma_*(\widehat{\mathcal{M}}))$. In what follows, we will introduce a Lefschetz class for u by studying $\hat{u}$ in $Hom_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}, \gamma_*(\widehat{\mathcal{M}}))$. Our construction generalizes analogous constructions in [KS].

[3] This is a minor abuse of terminology.

Lemma 3.1. *Let $\widehat{\mathcal{M}} \in D^b_{\mathrm{coh}}(\widehat{\mathcal{E}}_X)$. There is a natural morphism in $D^b_{\mathrm{coh}}(\widehat{\mathcal{E}}_{X\times X^a})$:*

$$\gamma_*(\widehat{\mathcal{M}}) \overset{L}{\boxtimes} D'_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}) \to \mathcal{C}^\gamma_X. \tag{2}$$

Proof. By [KS, Lemma 4.1.1], there is a natural morphism

$$\widehat{\mathcal{M}} \overset{L}{\boxtimes} D'_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}) \to \mathcal{C}_X.$$

Applying the functor $(\gamma \times 1)_*$ to the above morphism, we obtain the desired morphism

$$\gamma_*(\widehat{\mathcal{M}}) \overset{L}{\boxtimes} D'_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}) \to \mathcal{C}^\gamma_X.$$

□

Let $u \in Hom_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}, \gamma_*(\widehat{\mathcal{M}}))$. Consider the morphism

$$\begin{aligned} R\mathcal{H}om_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}, \gamma_*(\widehat{\mathcal{M}})) &\xleftarrow{\sim} D'_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}) \overset{L}{\otimes}_{\widehat{\mathcal{E}}_X} \gamma_*(\widehat{\mathcal{M}}) \\ &\cong \mathcal{C}_{X^a} \overset{L}{\otimes}_{\widehat{\mathcal{E}}_{X\times X^a}} \left(\gamma_*(\widehat{\mathcal{M}}) \overset{L}{\boxtimes} D'_{\widehat{\mathcal{E}}}(\widehat{\mathcal{M}})\right) \\ &\to \mathcal{C}_{X^a} \overset{L}{\otimes}_{\widehat{\mathcal{E}}_{X\times X^a}} \mathcal{C}^\gamma_X \xrightarrow{\delta_X^{-1}} \mathcal{H}\mathcal{H}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X). \end{aligned}$$

This defines a natural map

$$Hom_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}, \gamma_*(\widehat{\mathcal{M}})) \longrightarrow H^0_{\mathrm{supp}(\mathcal{M})}(X; \mathcal{H}\mathcal{H}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X)). \tag{3}$$

Definition 3.2. For an element $u \in Hom_{\mathcal{D}_M}(\mathcal{M}, \gamma_*(\mathcal{M}))$, define the Hochschild Lefschetz class $hh^\gamma(\mathcal{M}, u) \in H^0_{\mathrm{supp}(\mathcal{M})}(X; \mathcal{H}\mathcal{H}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X))$ to be the image of $\hat{u} \in Hom_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}, \gamma_*(\widehat{\mathcal{M}}))$ under the morphism (3).

Recall that $\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X)$ can be naturally identified with $RHom_{\widehat{\mathcal{E}}_{X\times X^a}}(\omega_X^{\otimes -1}, \mathcal{C}^\gamma_X)$ using the duality functor $D'_{\widehat{\mathcal{E}}_{X\times X^a}}$ and [KS, Theorem 2.5.7]. The following analog of [KS, Lemma 4.1.4] holds. We leave its proof to the interested reader.

Lemma 3.3. *Under the natural identification of $\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X)$ with $RHom_{\widehat{\mathcal{E}}_{X\times X^a}}(\omega_X^{\otimes -1}, \mathcal{C}^\gamma_X)$, the Hochschild Lefschetz class $hh^\gamma(\mathcal{M}, u)$ coincides with the following composite of morphisms*

$$\omega_X^{\otimes -1} \xrightarrow{\text{[KS, Lemma 4.1.1 (i)]}} \widehat{\mathcal{M}} \overset{L}{\boxtimes} D'_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}) \xrightarrow{\hat{u}\boxtimes id} \gamma_*(\widehat{\mathcal{M}}) \overset{L}{\boxtimes} D'_{\widehat{\mathcal{E}}_X}(\widehat{\mathcal{M}}) \xrightarrow{\textit{Lemma 3.1}} \mathcal{C}^\gamma_X\,.$$

Let X^γ be the submanifold[4] of X consisting of γ-fixed points and let $\iota : X^\gamma \hookrightarrow X$ be the inclusion. Let $\Omega^\bullet_{X^\gamma}$ be the (smooth) de Rham complex on X^γ (viewed as a complex of sheaves on X^γ). As in [PPT2, Section 3], one can construct a (distinguished) quasi-isomorphism with the quasi-isomorphism

$$\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X) \to \Omega^{(\dim(X^\gamma))-\bullet}$$

following a construction in [FFS, Section 4] and [EF, Section 2] (see also [FT]). Therefore,

Proposition 3.4. *The Hochschild homology $\mathcal{H}\mathcal{H}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X)$ is quasi-isomorphic (via a distinguished quasi-isomorphism) to $\iota_!(\mathbb{C}_{X^\gamma}[\dim(X^\gamma)])$.*

[4] X^γ is a disjoint union of embedded submanifolds possibly of different dimensions.

Thus one can define the *(microlocal) Lefschetz class* $\mu\mathrm{eu}^\gamma(\mathcal{M},u) \in H^{\dim X^\gamma}_{\mathrm{supp}(\mathcal{M})^\gamma}(X^\gamma,\mathbb{C})$ of u to be the image of $hh^\gamma(\mathcal{M},u)$ under the (distinguished) quasi-isomorphism

$$\mathcal{HH}(\widehat{\mathcal{E}}_X,\widehat{\mathcal{E}}^\gamma_X) \to \iota_!(\mathbb{C}_{X^\gamma}[\dim(X^\gamma)]).$$

When M is a point, γ acts on M trivially. Then $\mu\mathrm{eu}(\mathcal{M},u)$ is equal to the trace of u as an endomorphism of $\mathcal{M}$.

Remark 3.5. *One also has a class $eu(\mathcal{M},u) \in H^{2\dim M^\gamma}(M^\gamma,\mathbb{C})$. Its construction is completely analogous to that of $\mu eu^\gamma(\mathcal{M},u)$. When $\gamma = Id$ and when $\mathcal{M} = \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathcal{E}$ for some holomorphic vector bundle $\mathcal{E}$ on M, then $eu(\mathcal{M},u)$ is equal to the trace of u as an endomorphism of $\mathcal{O}_M \otimes^L_{\mathcal{D}_M} \mathcal{M}$ (see* [EF, FT, R]*).*

3.2. **Composition of Hochschild Lefschetz classes.** Consider three complex manifolds $M_i,\ i = 1,2,3$, and $X_i = T^*M_i,\ i = 1,2,3$. Let $\widehat{\mathcal{E}}_{X_1\times X_2^a}$ and $\widehat{\mathcal{E}}_{X_2\times X_3^a}$ be the sheaf of formal microdifferential operators on $X_i \times X_{i+1}^a,\ i = 1,2$. Assume that the group Γ acts on M_i and X_i holomorphically. Let p_{ij} be the canonical projection from $X_1 \times X_2 \times X_3$ to $X_i \times X_j$ for $1 \le i < j \le 3$. Also, let $d_i,\ i = 1,2,3$ denote the complex dimensions of the X_i. In this subsection, as in [KS], we implicitly identify $X = T^*M$ with its image in $X \times X$ under the embedding δ^γ_X whenever required. Also[5], to simplify notations, we sometimes denote $\widehat{\mathcal{E}}_{X_i}$ by $\widehat{\mathcal{E}}_i$: for example, $\otimes_{22^a}$ actually stands for $\otimes_{\widehat{\mathcal{E}}_{X_2\times X_2^a}}$.

Proposition 3.6. *There is a natural morphism*

$$\circ : Rp_{13!}\big(p_{12}^{-1}\mathcal{HH}(\widehat{\mathcal{E}}_{X_1\times X_2^a},\widehat{\mathcal{E}}^\gamma_{X_1\times X_2^a}) \overset{L}{\otimes} p_{23}^{-1}\mathcal{HH}(\widehat{\mathcal{E}}_{X_2\times X_3^a},\widehat{\mathcal{E}}^\gamma_{X_2\times X_3^a})\big) \longrightarrow \mathcal{HH}(\widehat{\mathcal{E}}_{X_1\times X_3^a},\widehat{\mathcal{E}}^\gamma_{X_1\times X_3^a}).$$

Proof. Following [KS], we will denote by $\widehat{\mathcal{E}}_{Z_i}$ the complex manifold $\widehat{\mathcal{E}}_{X_i\times X_i^a}$, and identify the Hochschild homology $\mathcal{HH}(\widehat{\mathcal{E}}_{X_i\times X_j^a},\widehat{\mathcal{E}}^\gamma_{X_i\times X_j^a})$ as follows:

$$\begin{aligned}
\mathcal{HH}(\widehat{\mathcal{E}}_{X_i\times X_j^a},\widehat{\mathcal{E}}^\gamma_{X_i\times X_j^a}) \quad &\cong \big(\mathcal{C}_{X_i^a} \overset{L}{\boxtimes} \mathcal{C}_{X_j}\big) \overset{L}{\otimes}_{\widehat{\mathcal{E}}_{Z_i\times Z_j^a}} \big(\mathcal{C}^\gamma_{X_i} \overset{L}{\boxtimes} \mathcal{C}^\gamma_{X_j^a}\big)\\
&\cong RHom_{\widehat{\mathcal{E}}_{Z_i\times Z_j^a}}\big(\omega^{\otimes -1}_{X_i} \overset{L}{\boxtimes} \omega^{\otimes -1}_{X_j^a}, \mathcal{C}^\gamma_{X_i} \overset{L}{\boxtimes} \mathcal{C}^\gamma_{X_j^a}\big)\\
&\cong RHom_{\widehat{\mathcal{E}}_{Z_i\times Z_j^a}}\big((\omega^{\otimes -1}_{X_i} \overset{L}{\boxtimes} \omega^{\otimes -1}_{X_j^a}) \underline{\overset{L}{\otimes}}_{\widehat{\mathcal{E}}_{X_j^a}} \omega_{X_j^a}, (\mathcal{C}^\gamma_{X_i} \overset{L}{\boxtimes} \mathcal{C}^\gamma_{X_j^a}) \underline{\overset{L}{\otimes}}_{\widehat{\mathcal{E}}_{X_j^a}} \omega_{X_j^a}\big)\\
&\cong RHom_{\widehat{\mathcal{E}}_{Z_i\times Z_j^a}}\big(\omega^{\otimes -1}_{X_i} \overset{L}{\boxtimes} \mathcal{C}_{X_j^a}, \mathcal{C}^\gamma_{X_i} \overset{L}{\boxtimes} \omega^\gamma_{X_j^a}\big).
\end{aligned}$$

As in [KS], let $S_{ij} := \omega^{\otimes -1}_{X_i} \overset{L}{\boxtimes} \mathcal{C}_{X_j^a}$, and let $K^\gamma_{ij} := \mathcal{C}^\gamma_{X_i} \overset{L}{\boxtimes} \omega^\gamma_{X_j^a}$. The above computation can be summarized as

$$\mathcal{HH}(\widehat{\mathcal{E}}_{X_i\times X_j^a},\widehat{\mathcal{E}}^\gamma_{X_i\times X_j^a}) \cong RHom_{\widehat{\mathcal{E}}_{Z_i\times Z_j^a}}(S_{ij},K^\gamma_{ij}). \tag{4}$$

We obtain the morphism

$$K^\gamma_{12} \underline{\overset{L}{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} K^\gamma_{23} \xrightarrow{\cong} (\mathcal{C}^\gamma_{X_1} \overset{L}{\boxtimes} \omega^\gamma_{X_2^a}) \underline{\overset{L}{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} (\mathcal{C}^\gamma_{X_2} \overset{L}{\boxtimes} \omega^\gamma_{X_3^a}) \longrightarrow p_{13}^{-1}(\mathcal{C}^\gamma_{X_1} \overset{L}{\boxtimes} \omega^\gamma_{X_3^a})[2d_2] = p_{13}^{-1}(K^\gamma_{13})[2d_2]. \tag{5}$$

For the last arrow in the above composition, note that $\omega^\gamma_{X_2^a} \underline{\overset{L}{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} \mathcal{C}^\gamma_{X_2}$ is naturally isomorphic to $(\gamma\times 1)_*(\omega_{X_2^a} \underline{\overset{L}{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} \mathcal{C}_{X_2})$. Also recall that the morphism $\omega_{X_2^a} \underline{\overset{L}{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} \mathcal{C}_{X_2} \to \delta_*\mathbb{C}_{X_2}[2d_2]$ is defined

[5] As did in [KS].

by [KS, Theorem 2.5.7]. Hence, one obtains a morphism $\omega^{\gamma}_{X_2^a} \overset{L}{\underline{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} \mathcal{C}^{\gamma}_{X_2} \to \delta^{\gamma}_{*}\mathbb{C}_{X_2}[2d_2]$ which induces the last arrow in the above composition. The morphism (5) induces, by adjunction, a morphism

$$Rp_{13!}(K^{\gamma}_{12} \overset{L}{\underline{\otimes}}_{\mathcal{E}_{Z_2}} K^{\gamma}_{23}) \to K^{\gamma}_{13}\,. \tag{6}$$

As explained in the proof of [KS, Proposition 4.2.1], there is a natural morphism

$$S_{13} \longrightarrow Rp_{13*}(S_{12} \overset{L}{\underline{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} S_{23}).$$

With the above two morphisms, we have natural morphisms

$$\begin{aligned}
&Rp_{13!}(p_{12}^{-1}\mathcal{HH}(\widehat{\mathcal{E}}_{X_1\times X_2^a}, \widehat{\mathcal{E}}^{\gamma}_{X_1\times X_2^a}) \overset{L}{\otimes} p_{23}^{-1}\mathcal{HH}(\widehat{\mathcal{E}}_{X_2\times X_3^a}, \widehat{\mathcal{E}}^{\gamma}_{X_2\times X_3^a}))\\
&\longrightarrow Rp_{13!}RHom_{\widehat{\mathcal{E}}_{Z_1\times Z_3^a}}\big(S_{12} \overset{L}{\underline{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} S_{23}, K^{\gamma}_{12} \overset{L}{\underline{\otimes}}_{\mathcal{E}_{Z_2}} K^{\gamma}_{23}\big)\\
&\longrightarrow RHom_{\widehat{\mathcal{E}}_{Z_1\times Z_3^a}}\big(Rp_{13*}(S_{12} \overset{L}{\underline{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} S_{23}), Rp_{13!}(K^{\gamma}_{12} \overset{L}{\underline{\otimes}}_{\widehat{\mathcal{E}}_{Z_2}} K^{\gamma}_{23})\big)\\
&\longrightarrow RHom_{\widehat{\mathcal{E}}_{Z_1\times Z_3^a}}(S_{13}, K^{\gamma}_{13}) \cong \mathcal{HH}(\widehat{\mathcal{E}}_{X_1\times X_3^a}, \widehat{\mathcal{E}}^{\gamma}_{X_1\times X_3^a}).
\end{aligned}$$

This proves the desired proposition. □

As a corollary, if $X_1 = X_3 = \mathrm{pt}$ and $X_2 = X$, then Proposition 3.6 defines a morphism

$$Ra_!\big(\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^{\gamma}_X) \overset{L}{\otimes} \mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^{\gamma}_X)\big) \to \mathbb{C}_{pt},$$

where $a : X \to pt$ is the natural map. By the adjunction formula, we have

Corollary 3.7. *Let $X_{\mathbb{R}}$ be the underlying real manifold of X. There is a canonical morphism $\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^{\gamma}_X) \overset{L}{\otimes} \mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^{\gamma}_X)) \to \omega^{top}_{X_{\mathbb{R}}}$, where $\omega^{top}_{X_{\mathbb{R}}}$ is the topological dualizing complex on $X_{\mathbb{R}}$ with coefficients in $\mathbb{C}$.*

Remark 3.8. *Let $HH_{\bullet}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^{\gamma}_X)$ denote the hypercohomology $H^{-\bullet}(X, \mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^{\gamma}_X))$. We remark that by integration, Corollary 3.7 defines a pairing on $HH_{\bullet}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^{\gamma}_X)$, which is a γ-equivariant generalization of the Fourier-Mukai pairing. We hope to discuss more about this pairing in a future publication.*

We recall that [KS, Definition 3.1.3] that for $\mathcal{K}_i \in D^b(\widehat{\mathcal{E}}_{X_i\times X_{i+1}^a})$ $(i = 1, 2)$,

$$\begin{aligned}
\mathcal{K}_1 \circ_{X_2} \mathcal{K}_2 &= Rp_{13!}(\mathcal{K}_1 \overset{L}{\underline{\otimes}}_{\widehat{\mathcal{E}}_2} \mathcal{K}_2) \in D^b(\widehat{\mathcal{E}}_{X_1\times X_3^a}),\\
\mathcal{K}_1 *_{X_2} \mathcal{K}_2 &= Rp_{13*}(\mathcal{K}_1 \overset{L}{\underline{\otimes}}_{\widehat{\mathcal{E}}_2} \mathcal{K}_2) \in D^b(\widehat{\mathcal{E}}_{X_1\times X_3^a}).
\end{aligned}$$

In what follows, we often simplify notations by writing $\circ_2$ for $\circ_{X_2}$ and $*_2$ for $*_{X_2}$.

We have the following generalization of [KS, Lemma 4.3.3].

Lemma 3.9. *Let $\gamma \in \Gamma$. Let $\mathcal{K}$ be a γ-equivariant element in $D^b_{coh}(\widehat{\mathcal{E}}_{X_1\times X_2^a})$. There is a natural morphism in $D^b(\widehat{\mathcal{E}}_{X_1\times X_1^a})$,*

$$\mathcal{K} \circ_2 \omega^{\gamma} \circ_2 D'_{\widehat{\mathcal{E}}}(\mathcal{K}) \longrightarrow \mathcal{C}^{\gamma}_{X_1}.$$

Proof. By Lemma 3.1, we have a morphism in $D^b(\widehat{\mathcal{E}}_{X_1\times X_2^a\times X_1^a\times X_2})$

$$\gamma_*(\mathcal{K})\underline{\boxtimes} D'_{\widehat{\mathcal{E}}}(\mathcal{K}) \longrightarrow \mathcal{C}^{\gamma}_{X_1\times X_2^a}.$$

Applying the functor $(-) \overset{L}{\otimes}_{X_2\times X_2^a} \omega_2^\gamma$, we obtain

$$\begin{aligned}&\big(\mathcal{K} \overset{L}{\boxtimes} D'_{\widehat{\mathcal{E}}}(\mathcal{K})\big) \otimes_{X_2\times X_2^a} \omega_{X_2}^\gamma \xrightarrow{(1)} \big(\gamma_*(\mathcal{K}) \overset{L}{\boxtimes} D'_{\widehat{\mathcal{E}}}(\mathcal{K})\big) \otimes_{X_2\times X_2^a} \omega_{X_2}^\gamma \\ &\xrightarrow{(2)} \mathcal{C}^\gamma_{X_1\times X_2^a} \overset{L}{\underline{\otimes}}_{X_2\times X_2^a} \omega_{X_2}^\gamma \xrightarrow{(3)} \mathcal{C}^\gamma_{X_1} \overset{L}{\underline{\otimes}} \mathbb{C}_{X_2}[2\dim(X_2)],\end{aligned}$$

Here, in arrow (1), we use the assumption that $\mathcal{K}$ is γ-equivariant, i.e. γ is a natural element in $Hom(\mathcal{K}, \gamma_*(\mathcal{K}))$; in arrow (2), we have used the morphism in Lemma 3.1; in arrow (3), we have used the natural isomorphism between $\mathcal{C}^\gamma_{X_2^a} \overset{L}{\underline{\otimes}}_{X_2\times X_2^a} \omega^\gamma_{X_2}$ and $\delta^\gamma_*\mathbb{C}_{X_2}[2d_2]$; this morphism is obtained by applying the functor $(\gamma\times 1)_*$ to the morphism from [KS, Theorem 2.5.7]. The desired morphism is induced by the above composition of morphisms via adjunction. □

For Λ a closed subset of X, let

$$HH_\Lambda(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X) := H^0(R\Gamma_\Lambda(X; \mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X))).$$

Let Λ_{12} and Λ_2 be closed subsets of $X_1\times X_2^a$ and X_2. Define $\Lambda_{12}\times_{X_2}\Lambda_2 \subset X_1\times X_2$ to be the fiber product of Λ_{12} and Λ_2 over X_2, and $\Lambda_{12}\circ\Lambda$ to be $p_1(\Lambda_{12}\times_{X_2}\Lambda_2)\subset X_1$. Given a γ-equivariant kernel $\mathcal{K}\in D^b_{\mathrm{coh}}(\widehat{\mathcal{E}}_{X_1\times X_2^a})$ with support Λ_{12}, we define the following map

$$\Phi_{\mathcal{K}} : HH_{\Lambda_2}(\widehat{\mathcal{E}}_{X_2}, \widehat{\mathcal{E}}^\gamma_{X_2}) \to HH_{\Lambda_{12}\circ\Lambda_2}(\widehat{\mathcal{E}}_{X_1}, \widehat{\mathcal{E}}^\gamma_{X_1})$$

via a sequence of compositions,

$$\begin{aligned}&HH_{\Lambda_2}(\widehat{\mathcal{E}}_{X_2}; \widehat{\mathcal{E}}^\gamma_{X_2}) \cong H^0(R\Gamma_{\Lambda_2} Hom_{X_2\times X_2^a}(\omega_2^{\otimes -1}, \mathcal{C}_2^\gamma)) \\ &\longrightarrow H^0\big(R\Gamma_{\Lambda_{12}\times_{X_2}\Lambda_2} Hom_{X_1\times X_1^a}(\mathcal{K} \overset{L}{\underline{\otimes}}_2 \omega_2^{\otimes -1}\circ_2 \omega_2 \circ_2 D'_{\widehat{\mathcal{E}}}\mathcal{K}, \mathcal{K} \overset{L}{\underline{\otimes}}_2 \mathcal{C}_2^\gamma \circ_2 \omega_2\circ_2 D'_{\widehat{\mathcal{E}}}\mathcal{K})\big) \\ &\longrightarrow H^0\big(R\Gamma_{\Lambda_{12}\circ\Lambda_2} Hom_{X_1\times X_1^a}(Rp_*(\mathcal{K} \overset{L}{\underline{\otimes}}_2 \omega_2^{\otimes -1}\circ_2 \omega_2 \circ_2 D'_{\widehat{\mathcal{E}}}\mathcal{K}), Rp_!(\mathcal{K} \overset{L}{\underline{\otimes}}_2 \mathcal{C}_2^\gamma \circ_2 \omega_2\circ_2 D'_{\widehat{\mathcal{E}}}\mathcal{K}))\big) \\ &\cong H^0(\Gamma_{\Lambda_{12}\circ\Lambda_2} Hom_{X_1\times X_1^a}(\mathcal{K} *_2 D'_{\widehat{\mathcal{E}}}\mathcal{K}, \mathcal{K}\circ_2 \omega_2^\gamma \circ_2 D'_{\widehat{\mathcal{E}}}(\mathcal{K}))) \\ &\longrightarrow H^0(R\Gamma_{\Lambda_{12}\circ\Lambda_2} Hom_{X_1\times X_1^a}(\omega^{\otimes -1}, \mathcal{C}_1^\gamma)) \cong HH_{\Lambda_{12}\circ\Lambda_2}(\widehat{\mathcal{E}}_{X_1}; \widehat{\mathcal{E}}^\gamma_{X_1}),\end{aligned}$$

where in the first arrow, we have applied the functor $\mathcal{L}\mapsto \mathcal{K}\overset{L}{\underline{\otimes}}_2 (\mathcal{L}\circ_2\omega_2\circ_2 D'_{\widehat{\mathcal{E}}}\mathcal{K})$, and in the last arrow we have used Lemma 3.9, and [KS, Lemma 4.3.3].

Let $f : X_2\to X_1$ be a γ-equivariant symplectic map. The graph Γ_f of f in $X_1\times X_2^a$ is a Lagrangian submanifold. Denote by $\mathcal{B}_{\Gamma_f}$ the holonomic D-module supported at Γ_f. It is easy to check from the property of f that $\mathcal{B}_{\Gamma_f}$ is γ-equivariant. By Definition 3.2, we can define $hh(f,\gamma) = hh^\gamma(\mathcal{B}_{\Gamma_f}, \gamma) \in H^0_{\Gamma_f}(X_1\times X_2; \mathcal{HH}(\widehat{\mathcal{E}}_{X_1\times X_2^a}, \widehat{\mathcal{E}}^\gamma_{X_1\times X_2^a}))$.

The proof of [KS, Lemma 4.3.4] may be generalized word for word to give the following result.

Proposition 3.10. *The following morphisms are equal,*

$$\Phi_{\mathcal{B}_{\Gamma_f}} = hh(f,\gamma)\circ : HH_{\Lambda_2}(\widehat{\mathcal{E}}_{X_2}, \widehat{\mathcal{E}}^\gamma_{X_2}) \to HH_{\Lambda_{12}\circ\Lambda_2}(\widehat{\mathcal{E}}_{X_1}, \widehat{\mathcal{E}}^\gamma_{X_1}).$$

Proof. Let α_2 be a class in $HH(\widehat{\mathcal{E}}_{X_2}, \widehat{\mathcal{E}}^\gamma_{X_2})$. By the isomorphism

$$\mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}^\gamma_X) \cong \delta_X^{-1} R\mathcal{H}om_{\widehat{\mathcal{E}}_{X\times X^a}}(D'_{\widehat{\mathcal{E}}_{X^a\times X}}(\mathcal{C}_{X^a}), \mathcal{C}^\gamma_X) \cong \delta_X^{-1} R\mathcal{H}om_{\widehat{\mathcal{E}}_{X\times X^a}}(\omega_X^{\otimes -1}, \mathcal{C}^\gamma_X),$$

we can regard α_2 as a morphism $\alpha_2 : \omega_{X_2}^{\otimes -1} \longrightarrow \mathcal{C}^\gamma_{X_2}$ in the derived category of sheaves of $\mathbb{C}$-vector spaces on $X_{22^a} := X_2\times X_2^a$. Similarly, we can regard the element $\alpha = hh(f,\gamma)$ in

$HH(\widehat{\mathcal{E}}_{X_1\times X_2^a}, \widehat{\mathcal{E}}^\gamma_{X_1\times X_2^a})$ as a morphism $\alpha : \omega^{\otimes -1}_{X_1\times X_2^a} \to \mathcal{C}^\gamma_{X_1\times X_2^a}$ in the derived category of sheaves on $X_{11^a22^a} := X_1 \times X_1^a \times X_2 \times X_2^a$. By Lemma 3.3, α is given by a composite of morphisms

$$\omega_{12^a}^{\otimes-1} \xrightarrow{\text{[KS, Lemma 4.1.1(i)]}} \mathcal{K} \overset{L}{\underline{\boxtimes}} D'\mathcal{K} \xrightarrow{\bar{\alpha}} \mathcal{C}^\gamma_{12^a}$$

in the derived category of sheaves on $X_{11^a22^a}$, where $\mathcal{B}_{\Gamma_f}$ is denoted by $\mathcal{K}$. The element $\Phi_{\mathcal{B}_{\Gamma_f}}(\alpha)$ is an element represented by the morphism

$$\begin{aligned}&\omega_1^{\otimes-1} \to \mathcal{K} *_2 D'_{\widehat{\mathcal{E}}}\mathcal{K} \to Rp_{1*}\big(\mathcal{K} \overset{L}{\underline{\otimes}}_2 (\omega_2^{\otimes-1} \circ_2 \omega_2 \circ_2 D'_{\widehat{\mathcal{E}}}\mathcal{K})\big)\\ &\xrightarrow{\alpha_2} Rp_{1!}\big(\mathcal{K} \overset{L}{\underline{\otimes}}_2 (\mathcal{C}_2^\gamma \circ_2 \omega_2 \circ_2 D'_{\widehat{\mathcal{E}}}\mathcal{K})\big) \to Rp_{1!}\big(\mathcal{K} \circ_2 \omega^\gamma \circ D'_{\widehat{\mathcal{E}}}\mathcal{K}\big) \xrightarrow{\text{Lemma 3.9}} \mathcal{C}^\gamma_{X_1}\end{aligned}$$

in the derived category of sheaves on X_1. The following commutative diagram in the category $D^b(\widehat{\mathcal{E}}_{11^a} \boxtimes \mathbb{C}_{X_2\times X_2^a})$ directly generalizes a subdiagram of a diagram appearing in the one in the proof of [KS, Lemma 4.3] (see [KS, Page 111]). The only genuine change in following diagram from the one in the proof of [KS, Lemma 4.3] is to change $\otimes_{22^a}\mathcal{C}_2$ to $\otimes_{22^a}\mathcal{C}_2^\gamma$. We also point out to the reader that the last row in the diagram below is written in a different (though equivalent) way than the corresponding row in [KS, Page 111] (modulo the above mentioned change from $\otimes_{22^a}\mathcal{C}_2$ to $\otimes_{22^a}\mathcal{C}_2^\gamma$).

$$\begin{tikzcd}
p^{-1}_{11^a}\omega_1^{\otimes -1} \arrow[d] & & \\
\big(\omega_1^{\otimes-1}\underline{\boxtimes}\mathcal{C}_{2^a}\big) \overset{L}{\underline{\otimes}}_{22^a} \omega_2^{\otimes -1} \arrow[r, "\alpha_2"] \arrow[d] & \big(\omega_1^{\otimes-1}\underline{\boxtimes}\mathcal{C}_{2^a}\big) \overset{L}{\underline{\otimes}}_{22^a} \mathcal{C}_2^\gamma \arrow[r] \arrow[d] & \big((\mathcal{K}\underline{\boxtimes}D'\mathcal{K})\circ_{2^a}\omega_{2^a}\big) \overset{L}{\underline{\otimes}}_{22^a}\mathcal{C}_2^\gamma \arrow[d, "\bar{\alpha}"] \\
(\mathcal{K}\underline{\boxtimes}D'_{\widehat{\mathcal{E}}}\mathcal{K}) \overset{L}{\underline{\otimes}}_{22^a} (\omega_2^{\otimes-1}\circ_2\omega_2) \arrow[r, "\alpha_2"] & (\mathcal{K}\underline{\boxtimes}D'_{\widehat{\mathcal{E}}}\mathcal{K}) \overset{L}{\underline{\otimes}}_{22^a} (\mathcal{C}_2^\gamma\circ_2\omega_2) \arrow[r, "\text{Lemma 3.9}"] & \mathcal{C}_1^\gamma\boxtimes\mathbb{C}_{X_2}[2\dim(X_2)]
\end{tikzcd}$$

By adjunction, the map from $p^{-1}_{11^a}\omega_1^{\otimes-1}$ to $\mathcal{C}_1^\gamma \boxtimes \mathbb{C}_{X_2}[2\dim(X_2)]$ via the composition of the upper row with the right column is $\alpha \circ \alpha_2$ while the map via the composition of the left column with the lower row is $\Phi_{\mathcal{B}_{\Gamma_f}}(\alpha_2)$. This gives the desired equality of morphisms. □

Remark 3.11. *It is interesting to compare Proposition 3.10 with* [G, Theorem 5.4]*, which is the direct image theorem for the Lefschetz class constructed in* [G]. *The integral transform* $\Phi_{\mathcal{B}_{\Gamma_f}}$ *in Proposition 3.10 corresponds to an honest morphism* $f : X_1 \to X_2$ *of complex manifolds in* [G]. *On the other hand, the holomorphic diffeomorphisms* γ_{X_1} *and* γ_{X_2} *that appear*[6] *in Proposition 3.10 correspond to to a pair of integral transforms, one on* X_1 *and the other on* X_2 *satisfying certain compatibility criteria with respect to* f*.*

It would be interesting to generalize the material in this and the previous subsection (Proposition 3.10 in particular) to the case when γ *acts on* X_1 *as well as* X_2 *by integral transforms rather than holomorphic diffeomorphisms. The approach here seriously utilizes the fact that* γ *acts by holomorphic diffeomorphisms, making such a generalization non-trivial. Such a generalization would yield a more general direct image theorem for the Hochschild Lefschetz class than* [G, Theorem

[6] γ_{X_1} (resp., γ_{X_2}) denotes the holomorphic diffeomorphism γ acting on X (resp., X_2)

5.4]. *Further, when combined with an understanding of trace densities, such a result would yield a (possibly simpler) approach to generalizations of* [G, Theorem 5.4] *itself.*

3.3. **Orbifold Hochschild and Chern class.** Let Q_X (and Q_M) be the orbifold defined by the quotient X/Γ (and M/Γ) for $X = T^*M$ and let $\mathfrak{q} : X \to Q_X$ be the canonical quotient map. Define a sheaf of algebras $\mathcal{A}$ on Q_X by

$$\mathcal{A}(U) := \widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U)) \rtimes \Gamma,$$

for any (sufficiently small) open subset $U \subset Q_X$. In the above definition, Γ acts on $\mathfrak{q}^{-1}(U)$, and therefore acts on the algebra $\widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U))$. $\widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U)) \rtimes \Gamma$ is the associated crossed product algebra.

Let $\mathcal{M}$ be a good Γ-equivariant coherent $\mathcal{D}_M$-module and $\widehat{\mathcal{M}}$ the corresponding Γ-equivariant $\widehat{\mathcal{E}}_X$-module. Define $\mathfrak{M}$ to be a sheaf on Q_X by

$$\mathfrak{M}(U) := \widehat{\mathcal{M}}(\mathfrak{q}^{-1}(U)),$$

for any (sufficiently small) open subset $U \subset Q_X$. On an open subset $U \subset Q_X$, both γ and $\widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U))$ naturally act on $\widehat{\mathcal{M}}(\mathfrak{q}^{-1}(U))$ with the appropriate commutation relation between these actions. This equips $\mathfrak{M}$ with a natural $\mathcal{A}$-module structure. It is not difficult to check that if $\mathcal{M}$ is a good coherent $\mathcal{D}_M$-module, $\mathfrak{M}$ is a good coherent $\mathcal{A}$-module. We apply the following theorem to construct the Hochschild class $hh^{\mathcal{A}}_{\mathrm{supp}(\mathcal{M})/\Gamma,i}(\mathfrak{M})$ and the cyclic class $ch^{\mathcal{A}}_{\mathrm{supp}(\mathcal{M})/\Gamma,i}(\mathfrak{M})$ of a perfect complex $\mathfrak{M}$ of $\mathcal{A}$-modules, where $\mathrm{supp}(\mathcal{M})/\Gamma$ is the support of $\mathfrak{M}$ in Q_X.

Theorem 3.12. *(*[BNT, Theorem 2.1.1.]*) Let Q be a topological space and Z a closed subset of Q. Let $\mathcal{A}$ be a sheaf of algebras on Q such that there is a global section $1 \in \Gamma(Q; \mathcal{A})$ which restricts to $1_{\mathcal{A}_x}$ for all $x \in Q$. Let $\mathcal{HC}^-(\mathcal{A})$ (resp., $\mathcal{HH}(\mathcal{A})$) be the sheaf of negative cyclic (resp., Hochschild) homologies[7] of $\mathcal{A}$. Denote by $K^i_Z(\mathcal{A})$ the i-th K-group of the category of perfect complexes of $\mathcal{A}$-modules which are acyclic outside Z. There exists the cyclic class $\mathrm{ch}^{\mathcal{A}}_{Z,i} : K^i_Z(\mathcal{A}) \to H^{-i}_Z(Q; \mathcal{HC}^-(\mathcal{A}))$ and the Hochschild class $hh^{\mathcal{A}}_{Z,i} : K^i_Z(\mathcal{A}) \to H^{-i}_Z(Q; \mathcal{HH}(\mathcal{A}))$ such that*

- *the composition*

$$K^i_Z(\mathcal{A}) \xrightarrow{\mathrm{ch}^{\mathcal{A}}_{Z,i}} H^{-i}_Z(Q; \mathcal{HC}^-(\mathcal{A})) \longrightarrow H^{-i}_Z(Q; \mathcal{HH}(\mathcal{A}))$$

coincides with $hh^{\mathcal{A}}_{Z,i}$;

- *for a perfect complex $\mathcal{F}^\bullet$ of $\mathcal{A}$-modules supported on Z the Hochschild class*

$$hh^{\mathcal{A}}_{Z,0}(\mathcal{F}^\bullet) \in H^0_Z(Q; \mathcal{HH}(\mathcal{A}))$$

coincides with the composition

$$k \xrightarrow{1 \mapsto id} \mathcal{RH}om_{\mathcal{A}}(\mathcal{F}^\bullet, \mathcal{F}^\bullet) \xleftarrow{\cong} (\mathcal{RH}om_{\mathcal{A}}(\mathcal{F}^\bullet, \mathcal{A}) \boxtimes \mathcal{F}^\bullet) \otimes^L_{\mathcal{A}\otimes\mathcal{A}^{op}} \mathcal{A} \xrightarrow{ev \otimes id} \mathcal{A} \otimes^L_{\mathcal{A}\otimes\mathcal{A}^{op}} \mathcal{A}.$$

Applying Theorem 3.12, for a good coherent Γ-equivariant $\mathcal{D}_M$-module $\mathcal{M}$, we have a well defined Hochschild class $hh^{\mathcal{A}}_{Z,0}(\mathfrak{M}) \in H^0_Z(Q_X; \mathcal{HH}(\mathcal{A}))$ and cyclic class $\mathrm{ch}^{\mathcal{A}}_{Z,0}(\mathfrak{M}) \in H^0_Z(Q_X; \mathcal{HC}^-(\mathcal{A}))$ where $Z = \mathrm{supp}(\mathcal{M})/\Gamma$ and $\mathcal{A}$ is the sheaf of crossed product algebras defined by $\mathcal{A}(U) := \widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U)) \rtimes \Gamma$ (for sufficiently smal open sets U in Q_X).

On Q_X, we can also consider the sheaf of algebras $\widehat{\mathcal{E}}_{Q_X}$ defined by

$$\widehat{\mathcal{E}}_{Q_X}(U) := \widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U))^\Gamma \quad \text{(for } U \text{ sufficiently small).}$$

[7]As in Section 2, we abuse terminology here: $\mathcal{HC}^-(\mathcal{A})$ and $\mathcal{HH}(\mathcal{A})$ are objects in the derived category of sheaves of $\mathbb{C}$-vector spaces on Q. Also, when $Q = X := T^*M$ as in Section 2 and when $\mathcal{A} = \widehat{\mathcal{E}}_X$, $\mathcal{HH}(\mathcal{A})$ as defined in [BNT] is isomorphic to $\mathcal{HH}(\mathcal{A})$ as defined in Section 2.

Here, $\widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U))^\Gamma$ is the space of Γ-invariant sections of $\widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U))$. Similarly, we consider the good coherent $\widehat{\mathcal{E}}_{Q_X}$-module $\widehat{\mathcal{M}}_{Q_X}$ defined by

$$\widehat{\mathcal{M}}_{Q_X}(U) := \widehat{\mathcal{M}}(\mathfrak{q}^{-1}(U))^\Gamma.$$

Applying Theorem 3.12 to $\widehat{\mathcal{E}}_{Q_X}$ and $\widehat{\mathcal{M}}_{Q_X}$, we obtain the Hochschild and cyclic classes

$$hh^{\widehat{\mathcal{E}}_{Q_X}}_{Z,0}(\widehat{\mathcal{M}}_{Q_X}) \in H^0_Z(Q_X; \mathcal{HH}(\widehat{\mathcal{E}}_{Q_X})) \text{ and } ch^{\widehat{\mathcal{E}}_{Q_X}}_{Z,0}(\widehat{\mathcal{M}}_{Q_X}) \in H^0_Z(Q_X; \mathcal{HC}(\widehat{\mathcal{E}}_{Q_X})),$$

where $Z = \mathrm{supp}(\mathcal{M})/\Gamma$.

Consider the global section

$$e = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma \in \Gamma(\mathcal{A})$$

of the sheaf $\mathcal{A}$. It is easy to check that e is a projection. Define a sheaf $\mathcal{V}$ of $\mathcal{A}$-$\widehat{\mathcal{E}}_{Q_X}$-bimodules by

$$\mathcal{V}(U) := \widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U)) \rtimes \Gamma e|_U,$$

and a sheaf $\mathcal{W}$ of $\widehat{\mathcal{E}}_{Q_X}$-$\mathcal{A}$-bimodules by

$$\mathcal{W}(U) := e|_U \widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U)) \rtimes \Gamma.$$

$\mathcal{V}$ and $\mathcal{W}$ are Morita equivalence bimodules between $\mathcal{A}$ and $\widehat{\mathcal{E}}_{Q_X}$. Under this Morita equivalence, $\widehat{\mathcal{M}}_{Q_X}$ corresponds to the sheaf $\mathfrak{M}$. With the explicit bimodules $\mathcal{V}$ and $\mathcal{W}$, we can easily check that under the Morita isomorphism between the Hochschild and cyclic homologies of $\widehat{\mathcal{E}}_{Q_X}$ and those of $\mathcal{A}$, the Hochschild and cyclic classes of $\widehat{\mathcal{M}}_{Q_X}$ are identified with those of $\mathfrak{M}$.

$$hh^{\mathcal{A}}_{Z,0}(\mathfrak{M}) = hh^{\widehat{\mathcal{E}}_{Q_X}}_{Z,0}(\widehat{\mathcal{M}}_{Q_X}) \in H^0_Z(Q_X; \mathcal{HH}(\mathcal{A})) \cong H^0_Z(Q_X; \mathcal{HH}(\widehat{\mathcal{E}}_{Q_X})),$$
$$\mathrm{ch}^{\mathcal{A}}_{Z,0}(\mathfrak{M}) = \mathrm{ch}^{\widehat{\mathcal{E}}_{Q_X}}_{Z,0}(\widehat{\mathcal{M}}_{Q_X}) \in H^0_Z(Q_X; \mathcal{HC}^-(\mathcal{A})) \cong H^0_Z(Q_X; \mathcal{HC}^-(\widehat{\mathcal{E}}_{Q_X})).$$

The Hochschild and cyclic homology of $\mathcal{A}$ is computed in [DE] and [NPPT]

$$\mu^{\mathcal{A}} : \mathcal{HH}(\mathcal{A}) \cong (\oplus_\gamma \mathbb{C}_{X^\gamma}[\dim(X^\gamma)])^\Gamma, \qquad \mu^{\mathcal{A}} : \mathcal{HC}^-(\mathcal{A}) \cong (\oplus_{\gamma,\bullet\geq 0}\mathbb{C}_{X^\gamma}[\dim(X^\gamma) - 2(\bullet)])^\Gamma,$$

where $\gamma\in\Gamma$ acts on $\oplus_\gamma \mathbb{C}_{X^\gamma}[\dim(X^\gamma)]$ mapping the α-component to the $\gamma\alpha\gamma^{-1}$-component.

Let IQ_X be the inertia orbifold associated to Q_X, defined by

$$IQ_X := (\sqcup_{\gamma\in\Gamma} X^\gamma)/\Gamma,$$

where $\gamma\in\Gamma$ acts on $\sqcup_\gamma X^\gamma$ by mapping (α, x) with $\alpha(x) = x$ to $(\gamma\alpha\gamma^{-1}, \gamma(x))$. Let $\iota_{IQ_X} : IQ_X \to Q_X$ be the natural map defined by forgetting the group element. Thus, we have

$$(\oplus_{\gamma\in\Gamma}\mathbb{C}_{X^\gamma}[\dim(X^\gamma)])^\Gamma = \iota_{IQ_X,*}\mathbb{C}[\dim(IQ_X)].$$

Definition 3.13. The *orbifold Euler class* $\mathrm{eu}_{Q_X}(\mathcal{M})$ (resp., the *orbifold Chern class* $\mathrm{ch}_{Q_X}(\mathcal{M})$) of a good Γ-equivariant coherent $\mathcal{D}_M$-module $\mathcal{M}$ is defined to be the images of $hh^{\mathcal{A}}_{Z,0}(\mathfrak{M})$ (resp., $\mathrm{ch}^{\mathcal{A}}_{Z,0}(\mathfrak{M})$) in $H^0_Z(IQ_X; \mathbb{C}[\dim(IQ_X)])$ (resp., $\oplus_{n\geq 0} H^0_Z(IQ_X; \mathbb{C}[\dim(IQ_X) - 2n])$).

Remark 3.14. *In* [BNT]*, the classes $hh^{\mathcal{A}}_{Z,0}$ and $\mathrm{ch}^{\mathcal{A}}_{Z,0}$ are called the Euler and the Chern class respectively. Here, we distinguish them from their images in the $H^\bullet_Z(IQ_X, \mathbb{C}[\dim(IQ_X)])$, which are closer to the classical Euler and Chern characters.*

In the remaining part of this section, we will explain the relation between the Hochschild Lefschetz class in Definition 3.2 and orbifold Hochschild class in Theorem 3.12.

We observe that $\sum_{\gamma\in\Gamma} hh^\gamma(\mathcal{M},\gamma) \in \bigoplus_{\gamma\in\Gamma} H^0_{\mathrm{supp}(\mathcal{M})}(X;\mathcal{HH}(\widehat{\mathcal{E}}_X,\widehat{\mathcal{E}}_X^\gamma))$ is invariant under the action of Γ on $\bigoplus_{\gamma\in\Gamma} H^0_{\mathrm{supp}(\mathcal{M})}(X;\mathcal{HH}(\widehat{\mathcal{E}}_X,\widehat{\mathcal{E}}_X^\gamma))$ induced by the conjugation action $\alpha\mapsto\gamma\alpha\gamma^{-1}$ of Γ on itself. Consider

$$\begin{aligned}\widetilde{hh}^{Q_X}_{Z,0}(\mathcal{M}) := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} hh^\gamma(\mathcal{M},\gamma) &\in \Big(\bigoplus_{\gamma\in\Gamma} H^0_{\mathrm{supp}(\mathcal{M})}(X;\mathcal{HH}(\widehat{\mathcal{E}}_X,\widehat{\mathcal{E}}_X^\gamma))\Big)^\Gamma \\ &\cong H^0_{\mathrm{supp}(\mathcal{M})}\Big(Q_X;\big(\bigoplus_{\gamma\in\Gamma}\mathcal{HH}(\widehat{\mathcal{E}}_X,\widehat{\mathcal{E}}_X^\gamma)\big)^\Gamma\Big).\end{aligned}$$

Here, by abuse of notation, we also use the symbol $\widehat{\mathcal{E}}_X$ to denote the sheaf $U\mapsto\widehat{\mathcal{E}}_X(\mathfrak{q}^{-1}(U))$ of algebras on the orbifold Q_X. Note that the sheaf $\widehat{\mathcal{E}}_X$ is a sheaf of algebras on Q_X with a (local) Γ-action and that $\widehat{\mathcal{E}}_{Q_X} = \widehat{\mathcal{E}}_X^\Gamma$. The Hochschild homology $H^0_Z(Q_X,\mathcal{HH}(\mathcal{A}))$ is naturally isomorphic to $\Big(\oplus_{\gamma\in\Gamma} H^0_{\mathrm{supp}(\mathcal{M})}(X;\mathcal{HH}(\widehat{\mathcal{E}}_X,\widehat{\mathcal{E}}_X^\gamma))\Big)^\Gamma$ (see e.g. [DE]). Identifying $\Big(\oplus_{\gamma\in\Gamma} H^0_{\mathrm{supp}(\mathcal{M})}(X;\mathcal{HH}(\widehat{\mathcal{E}}_X,\widehat{\mathcal{E}}_X^\gamma))\Big)^\Gamma$ with $H^0_Z(Q_X,\mathcal{HH}(\mathcal{A}))$ using this isomorphism, the following equality holds.

Theorem 3.15.

$$\widetilde{hh}^Q_{Z,0}(\mathcal{M}) = hh^{\mathcal{A}}_{Z,0}(\mathfrak{M}).$$

We shall now sketch the proof of Theorem 3.15, leaving details to the interested reader.

Sketch of proof. In what follows, $\widehat{\mathcal{E}} := \widehat{\mathcal{E}}_X$ is thought of as a sheaf of algebras on Q_X. Let $\gamma_*(\mathfrak{M})$ denote the sheaf $\mathfrak{M}$ on Q_X, whose $\widehat{\mathcal{E}}$-module structure is twisted by γ like $\gamma_*(M)$.

Define $L : \mathcal{H}om_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M},\mathfrak{M}) \to \big(\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\gamma_*(\mathfrak{M}))\big)^\Gamma$ by

$$L(F) := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma\circ F.$$

Here, $\gamma\circ F(m) = \gamma^{-1}(F(\gamma(m)))$ for $m\in\Gamma(U,\widehat{\mathcal{E}}_X)$. Define

$$\tilde{L} : \mathcal{H}om_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M},\widehat{\mathcal{E}}\rtimes\Gamma)\otimes_{\widehat{\mathcal{E}}\rtimes\Gamma}\mathfrak{M} \to \big(\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\otimes_{\widehat{\mathcal{E}}}\gamma_*(\mathfrak{M})\big)^\Gamma$$

by

$$\tilde{L}(F\otimes_{\widehat{\mathcal{E}}\rtimes\Gamma} m) := \frac{1}{|\Gamma|}\sum_\gamma\sum_\alpha F_\alpha\otimes\gamma^{-1}(\alpha(m)),$$

where $F_\alpha\in\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})$ is defined by $F=\sum_\alpha F_\alpha\otimes\alpha\in\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\otimes_{\widehat{\mathcal{E}}}(\widehat{\mathcal{E}}\rtimes\Gamma)$ and $\gamma^{-1}(\alpha(m))$ is viewed as a section of $\gamma_*(\mathfrak{M})$. Also recall that the action of an element $g\in\Gamma$ on $\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\otimes_{\widehat{\mathcal{E}}}\gamma_*(\mathfrak{M})$ takes a section of the form $F(-)\otimes m$ to $g(F(g^{-1}(-)))\otimes g.m$. The morphisms denoted by μ in the diagram below are the obvious "evaluation" maps:

$$\begin{array}{ccc}\mathcal{H}om_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M},\mathfrak{M}) & \xleftarrow{\ \mu\ } & \mathcal{H}om_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M},\widehat{\mathcal{E}}\rtimes\Gamma)\otimes_{\widehat{\mathcal{E}}\rtimes\Gamma}\mathfrak{M} \\ \Big\downarrow{\scriptstyle L} & & \Big\downarrow{\scriptstyle \tilde{L}} \\ \big(\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\gamma_*(\mathfrak{M}))\big)^\Gamma & \xleftarrow{\ \mu\ } & \big(\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\otimes_{\widehat{\mathcal{E}}}\gamma_*(\mathfrak{M})\big)^\Gamma\end{array}$$

It is straightforward to check that the following diagram commutes.

Define $\bar{L} : \big(\mathcal{H}om_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M}, \widehat{\mathcal{E}}\rtimes\Gamma)\boxtimes\mathfrak{M}\big)\otimes_{\mathcal{A}_{Q\times Q^a}}(\widehat{\mathcal{E}}\rtimes\Gamma) \to \Big(\big(\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\boxtimes\gamma_*(\mathfrak{M})\big)\otimes_{\widehat{\mathcal{E}}\otimes\widehat{\mathcal{E}}^a}\widehat{\mathcal{E}}\Big)^\Gamma$ by

$$\bar{L}\big(((F_\alpha\otimes\alpha)\boxtimes m)\otimes(d\otimes\beta)\big) := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}(F_\alpha\boxtimes\gamma^{-1}\alpha\beta(m))\otimes\alpha(d).$$

Here, $\gamma^{-1}\alpha\beta(m)$ is viewed as a section of $\gamma_*(\mathfrak{M})$. Also recall that the action of an element $g\in\Gamma$ on $\bigoplus_\gamma \mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\boxtimes\gamma_*(\mathfrak{M})$ takes a section $(F(-)\boxtimes m)\otimes f$ to $(g(F(g^{-1}(-)))\boxtimes g.m)\otimes g.f$. The following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}om_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M},\widehat{\mathcal{E}}\rtimes\Gamma)\otimes_{\widehat{\mathcal{E}}\rtimes\Gamma}\mathfrak{M} \arrow[d, "\tilde{L}"] & \big(\mathcal{H}om_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M},\widehat{\mathcal{E}}\rtimes\Gamma)\boxtimes\mathfrak{M}\big)\otimes_{\mathcal{A}_{Q\times Q^a}}(\widehat{\mathcal{E}}\rtimes\Gamma) \arrow[l, "\cong"'] \arrow[d, "\bar{L}"] \\
\big(\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\otimes_{\widehat{\mathcal{E}}}\gamma_*(\mathfrak{M})\big)^\Gamma & \Big(\big(\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\boxtimes\gamma_*(\mathfrak{M})\big)\otimes_{\widehat{\mathcal{E}}\otimes\widehat{\mathcal{E}}^a}\widehat{\mathcal{E}}\Big)^\Gamma \arrow[l, "\cong"']
\end{tikzcd}$$

Define $\hat{L} : (\widehat{\mathcal{E}}\rtimes\Gamma)\otimes_{\mathcal{A}_{Q\times Q^a}}(\widehat{\mathcal{E}}\rtimes\Gamma) \to \big(\bigoplus_{\gamma\in\Gamma}\gamma_*(\widehat{\mathcal{E}})\otimes\widehat{\mathcal{E}}\big)^\Gamma$ by

$$\hat{L}((e_0\otimes\alpha)\otimes(e_1\otimes\beta)) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma(e_0)\otimes\gamma\alpha(e_1).$$

Here, $\gamma(e_0)$ is viewed as a section of $(\gamma\alpha\beta\gamma^{-1})_*(\widehat{\mathcal{E}})$ and $\gamma\alpha(e_1)$ is viewed as a section of $\mathcal{E}$. Also recall that the action of an element $g\in\Gamma$ maps a section $e\otimes f$ of $\gamma_*(\widehat{\mathcal{E}})\otimes\mathcal{E}$ to the section $ge\otimes gf$ of $(g\gamma g^{-1})_*(\widehat{\mathcal{E}})\otimes\mathcal{E}$. We further have the following commutative diagram:

$$\begin{tikzcd}
\big(\mathcal{H}om_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M},\widehat{\mathcal{E}}\rtimes\Gamma)\boxtimes\mathfrak{M}\big)\otimes_{\mathcal{A}_{Q\times Q^a}}(\widehat{\mathcal{E}}\rtimes\Gamma) \arrow[r, "ev\otimes id"] \arrow[d, "\bar{L}"] & (\widehat{\mathcal{E}}\rtimes\Gamma)\otimes_{\mathcal{A}_{Q\times Q^a}}(\widehat{\mathcal{E}}\rtimes\Gamma) \arrow[d, "\hat{L}"] \\
\Big(\big(\bigoplus_{\gamma\in\Gamma}\mathcal{H}om_{\widehat{\mathcal{E}}}(\mathfrak{M},\widehat{\mathcal{E}})\boxtimes\gamma_*(\mathfrak{M})\big)\otimes_{\widehat{\mathcal{E}}\otimes\widehat{\mathcal{E}}^a}\widehat{\mathcal{E}}\Big)^\Gamma \arrow[r, "ev\otimes id"] & \big(\bigoplus_{\gamma\in\Gamma}\gamma_*(\widehat{\mathcal{E}})\otimes\widehat{\mathcal{E}}\big)^\Gamma
\end{tikzcd}$$

Combining the above three commutative diagrams gives us the desired theorem: indeed, the image of id $\in Hom_{\widehat{\mathcal{E}}\rtimes\Gamma}(\mathfrak{M},\mathfrak{M})$ in $H^0(Q_X,(\widehat{\mathcal{E}}\rtimes\Gamma)\overset{L}{\otimes}_{\mathcal{A}_{Q_X\times Q_X^a}}(\widehat{\mathcal{E}}\rtimes\Gamma))$ under the morphism induced by the upper horizontal arrows in the above three diagrams is $hh^{\mathcal{A}}_{Z,0}(\mathfrak{M})$, while the image of $\gamma\in Hom_{\widehat{\mathcal{E}}}(\mathfrak{M},\gamma_*(\mathfrak{M}))$ in $H^0(Q_X;\mathcal{HH}(\widehat{\mathcal{E}},\widehat{\mathcal{E}}^\gamma))$ under the morphism induced by the lower arrows in the above three diagrams is $hh^\gamma(\mathcal{M},\gamma)$. $\square$

Remark 3.16. *The orbifold Euler class in Definition 3.13 has a direct generalization to general orbifolds other than global quotient orbifolds, i.e. orbifolds of the form M/Γ. This generalization is obtained by working with the sheaf of invariant differential operators as explained in* [FT]. *Our orbifold Riemann-Roch theorem in the next section also generalizes to this setting. We will leave the details of this generalization to the reader.*

4. Euler class on an orbifold

In this section, we prove an orbifold Riemann-Roch theorem for the orbifold Euler class $\mathrm{eu}_{Q_X}(\mathcal{M})$ of a good Γ-equivariant coherent $\mathcal{D}_M$-module introduced in Definition 3.13. Our main strategy is to generalize the method developed by Bressler, Nest, and Tsygan [BNT] to orbifold setting.

4.1. **Deformation quantization.** Our strategy to compute the $\mathrm{eu}_{Q_X}(\mathcal{M})$ is to transfer the computation to a more flexible context: that of deformation quantization modules. Closely related to $\widehat{\mathcal{E}}_X$ and $\widehat{\mathcal{M}}$, is the (sheaf of) deformation quantization algebra(s) $\widehat{\mathcal{W}}_X(0)$ on $X = T^*M$ over the ring $\mathbb{C}[[\hbar]]$ constructed in [PS] modeled on (the sheaf of) negative order formal microdifferential operators. Let $\widehat{\mathcal{W}}_X$ be the localization of $\widehat{\mathcal{W}}(0)$ defined by

$$\widehat{\mathcal{W}}_X := \widehat{\mathcal{W}}(0) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar)).$$

The sheaves of algebras $\mathcal{D}_M$, $\widehat{\mathcal{E}}_X$, and $\widehat{\mathcal{W}}_X$ are naturally related to one another by the inclusions

$$\pi_M^{-1}\mathcal{D}_M \hookrightarrow \widehat{\mathcal{E}}_X \hookrightarrow \widehat{\mathcal{W}}_X.$$

Following [KS] we consider the functor

$$(\cdot)^W : Mod(\mathcal{D}_M) \to Mod(\widehat{\mathcal{W}}_X)\,,$$
$$\mathcal{M} \mapsto \mathcal{M}^W := \widehat{\mathcal{W}}_X \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{M}\,.$$

By [KS, Proposition 6.4.1], this functor is exact, faithful, and preserves properties such as coherence and goodness.

Note that the Lefschetz class of a good coherent $\mathcal{D}_M$-module with values in the Hochschild homology of $\widehat{\mathcal{W}}_X$ can be defined in exactly the same way as how the Lefschetz class of a good coherent $\mathcal{D}_M$-module with values in the Hochschild homology of $\widehat{\mathcal{E}}_X$ is defined in Section 3.

To be precise, given a good coherent $\mathcal{D}_M$-module $\mathcal{M}$, we consider the associated good coherent $\widehat{\mathcal{W}}_X$-module $\mathcal{M}^W$. As is explained in Section 2, γ_* defines a natural functor from $D^b(\widehat{\mathcal{W}}_X)$ (and $D^b_{\mathrm{coh}}(\widehat{\mathcal{W}}_X)$) to itself. Let $\mathcal{C}_X^W := \delta_{X,*}\widehat{\mathcal{W}}_X$, viewed as a $\widehat{\mathcal{W}}_{X\times X^a}$-module. Similarly, let $\mathcal{C}_X^{\gamma,W} := \delta^{\gamma}_{X,*}\widehat{\mathcal{W}}_X$. We have a natural morphism analogous to that of Lemma 3.1:

$$\gamma_*(\mathcal{M}^W) \overset{L}{\boxtimes} D'_{\widehat{\mathcal{W}}_X}(\mathcal{M}^W) \to \mathcal{C}_X^{\gamma,W}\,.$$

For $u \in Hom_{\mathcal{D}_M}(\mathcal{M}, \gamma_*(\mathcal{M}))$, the definition of the Hochschild Lefschetz class $hh^{\gamma,W}(\mathcal{M}, u)$ of a good coherent $\mathcal{D}_M$ module $\mathcal{M}$ is completely analogous to Definition 3.2. Indeed, $hh^{\gamma,W}(\mathcal{M}, u)$ is defined to be the image of $\hat{u}^W \in Hom_{\widehat{\mathcal{W}}_X}(\mathcal{M}^W, \gamma_*(\mathcal{M}^W))$ under the morphism induced on hypercohomologies by the following composite of morphisms:

$$\begin{aligned}
R\mathcal{H}om_{\widehat{\mathcal{W}}_X}(\mathcal{M}^W, \gamma_*(\mathcal{M}^W)) &\xleftarrow{\sim} D'_{\widehat{\mathcal{W}}_X}(\mathcal{M}^W) \overset{L}{\otimes}_{\widehat{\mathcal{W}}_X} \gamma_*(\mathcal{M}^W) \\
&\cong \mathcal{C}^W_{X^a} \overset{L}{\otimes}_{\widehat{\mathcal{W}}_{X\times X^a}} \left(\gamma_*(\mathcal{M}^W) \overset{L}{\boxtimes} D'_{\widehat{\mathcal{W}}}(\mathcal{M}^W)\right) \\
&\to \mathcal{C}^W_{X^a} \overset{L}{\otimes}_{\widehat{\mathcal{W}}_{X\times X^a}} \mathcal{C}_X^{\gamma,W} = \mathcal{HH}(\widehat{\mathcal{W}}_X, \widehat{\mathcal{W}}_X^{\gamma}).
\end{aligned}$$

One can similarly provide definitions of $\mu eu^{\gamma,W}(\mathcal{M}, u)$, $\mathrm{eu}^W_{Q_X}(\mathcal{M})$, and $\mathrm{ch}^W_{Q_X}(\mathcal{M})$ that are completely analogous to the corresponding definitions in Section 3.

Recall that the support of $\widehat{\mathcal{M}} := \widehat{\mathcal{E}} \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{M}$ in X is called the characteristic variety of $\mathcal{M}$ and denoted by $\mathrm{char}(\mathcal{M})$. The following Lemma is a direct generalization of [KS, Lemma 6.5.1] to the γ twisted setting. Let $\iota : X^{\gamma} \to X$ be as in Section 3.

Proposition 4.1. *There is a natural trace density isomorphism*

$$\mathcal{HH}(\widehat{\mathcal{W}}_X, \widehat{\mathcal{W}}_X^\gamma) \xrightarrow{\mu^{\widehat{\mathcal{W}}}} \iota_! \mathbb{C}_{X^\gamma}((\hbar))[\dim(X^\gamma)]$$

in the derived category of sheaves of $\mathbb{C}((\hbar))$-vector spaces on X such that the diagram following commutes:

$$\begin{tikzcd} \mathcal{HH}(\widehat{\mathcal{E}}_X, \widehat{\mathcal{E}}_X^\gamma) \arrow[r, "\mu^{\widehat{\mathcal{E}}}"] \arrow[d] & \iota_! \mathbb{C}_{X^\gamma}[\dim(X^\gamma)] \arrow[d] \\ \mathcal{HH}(\widehat{\mathcal{W}}_X, \widehat{\mathcal{W}}_X^\gamma) \arrow[r, "\mu^{\widehat{\mathcal{W}}}"] & \iota_! \mathbb{C}((\hbar))_{X^\gamma}[\dim(X^\gamma)] \end{tikzcd} .$$

Therefore, using the natural map from $H^{\dim(X^\gamma)}_{char(\mathcal{M})^\gamma}(X; \mathbb{C}_{X^\gamma})$ to $H^{\dim(X^\gamma)}_{char(\mathcal{M})^\gamma}(X; \mathbb{C}_{X^\gamma}((\hbar)))$ to identify $H^{\dim(X^\gamma)}_{char(\mathcal{M})^\gamma}(X; \mathbb{C}_{X^\gamma})$ with its image in $H^{\dim(X^\gamma)}_{char(\mathcal{M})^\gamma}(X; \mathbb{C}_{X^\gamma}((\hbar)))$, one obtains the following identities for a good Γ-equivariant coherent $\mathcal{D}_M$ module $\mathcal{M}$.

$$hh^\gamma(\mathcal{M}, \gamma) = hh^{\gamma, W}(\mathcal{M}, \gamma),\ \mathrm{eu}_{Q_X}(\mathcal{M}) = \mathrm{eu}^W_{Q_X}(\mathcal{M}),$$
$$\mu eu^\gamma(\mathcal{M}, \gamma) = \mu eu^{\gamma, W}(\mathcal{M}, \gamma),\ \mathrm{ch}_{Q_X}(\mathcal{M}) = \mathrm{ch}^W_{Q_X}(\mathcal{M}).$$

4.2. **Orbifold Riemann-Roch theorem.** In this subsection, we describe the geometric formula for the orbifold Euler class $\mathrm{Eu}_Q(\mathcal{M})$ of a good Γ-equivariant coherent $\mathcal{D}_M$ module $\mathcal{M}$.

We recall some geometry of the orbifold $Q_X = X/\Gamma$. Note that X^γ may have several components with different dimensions, but each component of X^γ is a submanifold of X. Consider a vector bundle V on X^γ equipped with a γ action on each fiber. Let R^V be the curvature of a connection on V. Define $\mathrm{ch}_\gamma(V)$ to be

$$\mathrm{ch}_\gamma(V) := \mathrm{tr}\left(\gamma \exp\left(\frac{R^V}{2\pi\sqrt{-1}}\right)\right) \in H^{even}(X^\gamma; \mathbb{C}).$$

Over each component of X^γ, let N^γ be the normal bundle of X^γ to X. Observe that γ acts on fibers of N^γ and $\wedge^\bullet N^\gamma$. Define

$$\mathrm{eu}_\gamma(N^\gamma) := \sum_\bullet (-1)^\bullet \mathrm{ch}_\gamma(\wedge^\bullet N^\gamma) = \det\left(1 - \gamma^{-1}\exp\left(\frac{-R^\perp}{2\pi\sqrt{-1}}\right)\right),$$

where $R^\perp$ is the curvature of a connection on N^γ. The group Γ naturally acts on $\sqcup_\gamma X^\gamma$: $\gamma \in \Gamma$ maps $x \in X^\alpha$ to $\gamma(x) \in X^{\gamma\alpha\gamma^{-1}}$. It is straightforward to see that

$$\mathrm{eu}_Q(N) := \sum_{\gamma\in\Gamma} \mathrm{eu}_\gamma(N^\gamma) \in \bigoplus_{\gamma\in\Gamma} H^{even}_{\mathrm{char}(\mathcal{M})^\gamma}(X^\gamma; \mathbb{C})$$

is invariant under the Γ action on the above direct sum of the cohomology groups.

Given a good Γ-equivariant coherent $\mathcal{D}_M$-module $\mathcal{M}$, we consider

$$\widehat{\mathcal{M}} := \widehat{\mathcal{E}}_X \otimes_{\pi_M^{-1}\mathcal{D}_M} \pi_M^{-1}\mathcal{M},$$

equipped with a natural filtration, whose support is denoted by $\mathrm{char}(\mathcal{M})$. The sheaf $\widehat{\mathcal{E}}_X$ has a natural filtration $\{\mathcal{F}^\bullet \widehat{\mathcal{E}}_X\}$ by the order $\bullet$ of an operator. Let $\mathrm{Gr}\widehat{\mathcal{E}}_X$ (resp., $\mathrm{Gr}\widehat{\mathcal{M}}$) denote the associated graded algebra and module of $\widehat{\mathcal{E}}_X$ (resp., $\widehat{\mathcal{M}}$). Define

$$\widetilde{\mathrm{Gr}}(\widehat{\mathcal{M}}) := \mathcal{O}_X \otimes_{\mathrm{Gr}\widehat{\mathcal{E}}_X} \mathrm{Gr}\widehat{\mathcal{M}}.$$

The symbol $\sigma_{\mathrm{char}(\mathcal{M})}(\mathcal{M})$ is the element in the Γ-equivariant K-theory $K^{top,\Gamma}_{\mathrm{char}(\mathcal{M})}(T^*M)$ defined by $\widetilde{\mathrm{Gr}}(\widehat{\mathcal{M}})$. Restricted to X^γ, $\sigma_{\mathrm{char}(\mathcal{M})}(\mathcal{M}|_{X^\gamma})$ defines a K-theory element on X^γ inheriting a γ-action.

Applying $\mathrm{ch}_\gamma(-)$ to this element, one defines an element $\mathrm{ch}_\gamma(\sigma_{\mathrm{char}(\mathcal{M})}(\mathcal{M}|_{X^\gamma}))$ in $H^{even}_{\mathrm{char}(\mathcal{M})^\gamma}(X^\gamma)$. As $\mathcal{M}$ is Γ-equivariant, one easily checks that the collection

$$\mathrm{ch}_Q(\sigma_{\mathrm{char}(\mathcal{M})}(\mathcal{M})) := \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\mathrm{ch}_\gamma\big(\sigma_{\mathrm{char}(\mathcal{M})}(\mathcal{M}|_{X^\gamma})\big) \in \bigoplus_{\gamma\in\Gamma} H^{even}_{\mathrm{char}(\mathcal{M})^\gamma}(X^\gamma;\mathbb{C})$$

is invariant under the Γ action on the above direct sum of the cohomology groups.

Define $\mathrm{char}_Q(\mathcal{M}) \subset IQ_X$ to be the quotient $(\sqcup_{\gamma\in\Gamma}\mathrm{char}(\mathcal{M})^\gamma)/\Gamma$, where Γ acts on $\sqcup_{\gamma\in\Gamma}\mathrm{char}(\mathcal{M})^\gamma$ via its action on $\sqcup_{\gamma\in\Gamma}X^\gamma$. We are now ready to state the main theorem of this paper.

Theorem 4.2. *(Orbifold Riemann-Roch) Let Q_M be the quotient of M by Γ. For a good Γ-equivariant coherent $\mathcal{D}_M$ module $\mathcal{M}$, we have*

$$\mathrm{eu}_Q(\mathcal{M}) = \frac{1}{m}\Big(\mathrm{ch}_Q(\sigma_{char(\mathcal{M})}(\mathcal{M}))\wedge \mathrm{eu}_Q(N)\wedge \pi^* Td_{IQ_M}\Big)_{\dim(IQ_X)}$$

as a cohomology class in $H^{\dim(IQ_X)}_{char_Q(\mathcal{M})}(IQ_X;\mathbb{C})$. Here, Td_{IQ_M} is the Todd class of the orbifold IQ_M defined by

$$Td_{IQ_M} := \mathrm{tr}\left(\frac{\frac{R}{2\pi\sqrt{-1}}}{1-\exp^{-\frac{R}{2\pi\sqrt{-1}}}}\right),$$

where R is the curvature of a connection on the tangent bundle TIQ_M, $\pi : IQ_X \to IQ_M$ is the natural projection, and m denotes the locally constant function on IQ measuring the size of the isotropy group.

Remark 4.3. *The wedge product of differential forms on IQ_X used in Theorem 4.2 is the wedge product on each component of IQ_X.*

The proof of Theorem 4.2 occupies the next two subsections. Our basic idea is to generalize the Bressler-Nest-Tsygan proof in [BNT] to the Γ-equivariant setting.

4.3. **Rees construction.** We consider the Rees ring $\mathcal{R}\widehat{\mathcal{E}}_X$ associated to the filtration $\{\mathcal{F}^\bullet\widehat{\mathcal{E}}_X\}$ of $\widehat{\mathcal{E}}_X$:

$$\mathcal{R}\widehat{\mathcal{E}}_X := \bigoplus_p \hbar^p\mathcal{F}^p\widehat{\mathcal{E}}_X.$$

We list a few well-known properties of $\mathcal{R}\widehat{\mathcal{E}}_X$ without proofs.

Proposition 4.4. *(c.f.* [BNT]*)*

(1) *Let $\mathrm{Gr}\widehat{\mathcal{E}}_X$ be the associated graded ring of $\widehat{\mathcal{E}}_X$ with respect to the filtration $\mathcal{F}^\bullet\widehat{\mathcal{E}}_X$. There are natural algebra homomorphisms*

$$\sigma^{\mathcal{R}\widehat{\mathcal{E}}} : \mathcal{R}\widehat{\mathcal{E}}_X \xrightarrow{\mathrm{Gr}} \mathrm{Gr}\widehat{\mathcal{E}}_X \xrightarrow{\iota} \mathcal{O}_X.$$

(2) *(*[BNT, I, Proposition 4.5.1]*) There is a natural flat embedding by mapping $\hbar\xi$ to ξ along the fiber direction of T^*M,*

$$i^{\mathcal{R}\widehat{\mathcal{E}}} : \mathcal{R}\widehat{\mathcal{E}}_X \longrightarrow \widehat{\mathcal{W}}_X(0).$$

(3) *Define $\mathcal{R}\widehat{\mathcal{E}}_X[\hbar^{-1}] := \mathcal{R}\widehat{\mathcal{E}}_X \otimes_{\mathbb{C}[[\hbar]]}\mathbb{C}((\hbar))$. There is a natural identification*

$$\mathcal{R}\widehat{\mathcal{E}}_X[\hbar^{-1}] \cong \widehat{\mathcal{E}}_X((\hbar)) := \widehat{\mathcal{E}}_X\otimes_{\mathbb{C}}\mathbb{C}((\hbar)).$$

(4) *Given a $\widehat{\mathcal{E}}_X$-module $\widehat{\mathcal{M}}$ with a filtration $\mathcal{F}^\bullet\widehat{\mathcal{M}}$, define an $\mathcal{R}\widehat{\mathcal{E}}_X$-module by*

$$\mathcal{R}\widehat{\mathcal{M}} := \bigoplus_p \hbar^p\mathcal{F}^p\widehat{\mathcal{M}}.$$

One has the following natural isomorphisms of $\mathcal{O}_X$-modules,

$$\mathcal{R}\widehat{\mathcal{M}} \otimes_{\mathcal{R}\widehat{\mathcal{E}}_X} \mathcal{R}\widehat{\mathcal{E}}_X[\hbar^{-1}] \quad \cong \quad \widehat{\mathcal{M}} \otimes_{\widehat{\mathcal{E}}_X} \widehat{\mathcal{E}}_X[\hbar^{-1}, \hbar], \tag{7}$$

$$\mathcal{R}\widehat{\mathcal{M}} \otimes_{\mathcal{R}\widehat{\mathcal{E}}_X} \mathcal{O}_X \quad \cong \quad \mathrm{Gr}\widehat{\mathcal{M}} \otimes_{\mathrm{Gr}\widehat{\mathcal{E}}_X} \mathcal{O}_X. \tag{8}$$

In addition, when a finite group Γ acts on M, the same results hold for $\widehat{\mathcal{E}}_X \rtimes \Gamma$, $\widehat{\mathcal{W}}_X(0) \rtimes \Gamma$, $\mathcal{R}\widehat{\mathcal{E}}_X \rtimes \Gamma$, $\mathrm{Gr}\widehat{\mathcal{E}}_X \rtimes \Gamma$, $\mathcal{O}_X \rtimes \Gamma$ and a Γ-equivariant $\widehat{\mathcal{E}}_X$-module $\widehat{\mathcal{M}}$.

By Proposition 4.4 (3), we have natural morphisms

$$\iota^{\hbar^{-1}, \mathcal{R}\widehat{\mathcal{E}}} : \mathcal{R}\widehat{\mathcal{E}}_X \longrightarrow \widehat{\mathcal{E}}_X[\hbar^{-1}, \hbar] \longleftarrow \widehat{\mathcal{E}}_X : \iota^{\hbar^{-1}, \widehat{\mathcal{E}}}. \tag{9}$$

Following the idea developed in Sec. 3.3, when a finite group Γ group acts on a manifold M and therefore also $X = T^*M$, we view $\widehat{\mathcal{E}}_{Q_X} := \widehat{\mathcal{E}}_X \rtimes \Gamma$, $\mathcal{R}\widehat{\mathcal{E}}_{Q_X} := \mathcal{R}\widehat{\mathcal{E}}_X \rtimes \Gamma$, $\mathrm{Gr}\widehat{\mathcal{E}}_{Q_X} := \mathrm{Gr}\widehat{\mathcal{E}}_X \rtimes \Gamma$, $\mathcal{O}_{Q_X} := \mathcal{O}_X \rtimes \Gamma$, $\widehat{\mathcal{W}}(0)_{Q_X} := \widehat{\mathcal{W}}(0)_X \rtimes \Gamma$, and $\widehat{\mathcal{W}}_{Q_X} := \widehat{\mathcal{W}}_X \rtimes \Gamma$ as sheaves of algebras over the orbifold $Q_X = X/\Gamma$. And similarly a Γ-equivariant $\widehat{\mathcal{E}}_X$-module $\widehat{\mathcal{M}}$ is viewed as a sheaf of $\widehat{\mathcal{E}}_{Q_X} := \widehat{\mathcal{E}}_X \rtimes \Gamma$-module $\widehat{\mathcal{M}}_{Q_X}$ over Q_X.

A crucial observation is that Theorem 3.12 applies to the sheaves of algebras introduced above and defines Chern character maps on the corresponding K-groups of perfect complexes of modules. These Chern characters are denoted by $\mathrm{ch}^A(-)$ with A being relevant sheaves of algebras. We apply Proposition 4.4 to study the Chern character of $\widehat{\mathcal{M}}_{Q_X}$:

Proposition 4.5.

$$\sigma_*^{\mathcal{R}\widehat{\mathcal{E}}} \mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X}) = \iota_* \mathrm{ch}^{\mathrm{Gr}\widehat{\mathcal{E}}}(\mathrm{Gr}\widehat{\mathcal{M}}_{Q_X}) = \mathrm{ch}^{\mathcal{O}}(\widehat{\mathcal{M}}_{Q_X} \otimes_{\widehat{\mathcal{E}}_{Q_X}} \mathcal{O}_{Q_X})$$

Proof. This is a direct corollary of Proposition 4.4 (1), and Eq. (8). □

Proposition 4.6.

$$\iota_*^{\hbar^{-1}, \mathcal{R}\widehat{\mathcal{E}}} \mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X}) = \mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}[\hbar^{-1}]}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X} \otimes_{\mathcal{R}\widehat{\mathcal{E}}_{Q_X}} \mathcal{R}\widehat{\mathcal{E}}_{Q_X}[\hbar^{-1}]) = \iota_*^{\hbar^{-1}, \widehat{\mathcal{E}}}(\mathrm{ch}^{\widehat{\mathcal{E}}}(\widehat{\mathcal{M}}_{Q_X})).$$

Proof. This is a direct corollary of Eq. (7), (9), and Proposition 4.4, (3). □

There is a natural map $\sigma^{\widehat{\mathcal{W}}(0)} : \widehat{\mathcal{W}}_X(0) \to \mathcal{O}_X$ defined by taking the quotient by the two sided (sheaf of) ideal(s) generated by $\hbar$. It is easy to check that

$$\sigma^{\mathcal{R}\widehat{\mathcal{E}}} = \sigma^{\widehat{\mathcal{W}}(0)} \circ i^{\mathcal{R}\widehat{\mathcal{E}}} : \mathcal{R}\widehat{\mathcal{E}}_{Q_X} \longrightarrow \mathcal{O}_{Q_X}. \tag{10}$$

Proposition 4.7.

$$\sigma_*^{\mathcal{R}\widehat{\mathcal{E}}}(\mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X})) = \sigma_*^{\widehat{\mathcal{W}}(0)} \circ i_*^{\mathcal{R}\widehat{\mathcal{E}}}(\mathrm{ch}^{\widehat{\mathcal{E}}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X})) = \mathrm{ch}^{\mathcal{O}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X} \otimes_{\widehat{\mathcal{E}}_{Q_X}} \mathcal{O}_{Q_X}).$$

Proof. This is a direct corollary of Eq. (10). □

Proposition 4.8. *The following diagram commutes*

$$\begin{tikzcd}
\widehat{\mathcal{W}}(0)_{Q_X} \arrow[r, "\iota^{\hbar^{-1}, \widehat{\mathcal{W}}}"] & \widehat{\mathcal{W}}_{Q_X} & \\
\mathcal{R}\widehat{\mathcal{E}}_{Q_X} \arrow[u, "i^{\mathcal{R}\widehat{\mathcal{E}}}"] \arrow[r, "\iota^{\hbar^{-1}, \mathcal{R}\widehat{\mathcal{E}}}"] & \mathcal{R}\widehat{\mathcal{E}}_{Q_X}[\hbar^{-1}] \arrow[u, "i^{\mathcal{R}\widehat{\mathcal{E}}[\hbar^{-1}]}"] & \widehat{\mathcal{E}}_{Q_X} \arrow[l, "\iota^{\hbar^{-1}, \widehat{\mathcal{E}}}"'] \arrow[ul, "i^{\widehat{\mathcal{E}}}"']
\end{tikzcd},$$

where $\iota^{\hbar^{-1}, \widehat{\mathcal{W}}}$ is the natural inclusion map $\widehat{\mathcal{W}}(0)_{Q_X} \hookrightarrow \widehat{\mathcal{W}}_{Q_X}$, and $i^{\mathcal{R}\widehat{\mathcal{E}}[\hbar^{-1}]}$ is the natural extension of $i^{\mathcal{R}\widehat{\mathcal{E}}}$.

Proof. This is a straightforward verification using the definitions. $\square$

We denote the hypercohomology $H^{-\bullet}(IQ_X;\mathcal{HC}^{per}(\widehat{\mathcal{W}}_{Q_X}))$ by $HC^{per}_{\bullet}(\widehat{\mathcal{W}}_{Q_X})$, where $\mathcal{HC}^{per}(-)$ is the sheaf of periodic cyclic homology. Similar notation is used for other sheaves of algebras on Q_X (and for other versions of cyclic homology). Note that for any sheaf $\mathcal{A}$ of algebras on Q_X (or on any topological space for that matter), there is a natural map $\mathcal{HC}^{-}(\mathcal{A}) \to \mathcal{HC}^{per}(\mathcal{A})$ in the derived category of sheaves of $\mathbb{C}$-vector spaces on Q_X. Hence, one may view $\mathrm{ch}^{\mathcal{A}}_{Z,i}$ (see Theorem 3.12) as a map to $HC^{per}_0(\mathcal{A})$. The following proposition is a direct corollary of Proposition 4.8.

Proposition 4.9. *The following diagram commutes.*

$$\begin{tikzcd}
 & K^0_\Lambda(\widehat{\mathcal{W}}(0)_{Q_X}) \arrow[rr, "\iota_*^{\hbar^{-1},\widehat{\mathcal{W}}}"] \arrow[dl, "\mathrm{ch}^{\widehat{\mathcal{W}}(0)}"'] & & K^0_\Lambda(\widehat{\mathcal{W}}_{Q_X}) \arrow[dl, "\mathrm{ch}^{\widehat{\mathcal{W}}}"'] & \\
HC^{per}_0(\widehat{\mathcal{W}}(0)_{Q_X}) \arrow[rr, "\iota_*^{\hbar^{-1},\widehat{\mathcal{W}}}"] & & HC^{per}_0(\widehat{\mathcal{W}}_{Q_X}) & & \\
 & K^0_\Lambda(\mathcal{R}\widehat{\mathcal{E}}_{Q_X}) \arrow[uu, dotted, "i_*^{\mathcal{R}\widehat{\mathcal{E}}}"'] \arrow[rr, dotted, "\iota_*^{\hbar^{-1},\mathcal{R}\widehat{\mathcal{E}}}"] \arrow[dl, dotted, "\mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}}"'] & & K^0_\Lambda(\mathcal{R}\widehat{\mathcal{E}}_{Q_X}[\hbar^{-1}]) \arrow[uu, dotted, "i_*^{\mathcal{R}\widehat{\mathcal{E}}[\hbar^{-1}]}"'] \arrow[dl, dotted, "\mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}[\hbar^{-1}]}"'] & K^0_\Lambda(\widehat{\mathcal{E}}_{Q_X}) \arrow[l, dotted, "\iota_*^{\hbar^{-1},\widehat{\mathcal{E}}}"'] \arrow[uul, "i_*^{\widehat{\mathcal{E}}}"'] \arrow[dl, "\mathrm{ch}^{\widehat{\mathcal{E}}}"] \\
HC^{per}_0(\mathcal{R}\widehat{\mathcal{E}}_{Q_X}) \arrow[uu, "i_*^{\mathcal{R}\widehat{\mathcal{E}}}"] \arrow[rr, "\iota_*^{\hbar^{-1},\mathcal{R}\widehat{\mathcal{E}}}"] & & HC^{per}_0(\mathcal{R}\widehat{\mathcal{E}}_{Q_X}[\hbar^{-1}]) \arrow[uu, "i_*^{\mathcal{R}\widehat{\mathcal{E}}[\hbar^{-1}]}"] & HC^{per}_0(\widehat{\mathcal{E}}_{Q_X}) \arrow[l, "\iota_*^{\hbar^{-1},\widehat{\mathcal{E}}}"'] \arrow[uul, "i_*^{\widehat{\mathcal{E}}}"] &
\end{tikzcd}$$

4.4. **Proof of Theorem 4.2.** The following is a reformulation of [PPT2, Theorem 5.13].

Theorem 4.10. *(*[PPT2, Theorem 5.13]*) Let u be the parameter in the definition of cyclic homology, and Q_M (resp., Q_X) be the quotient of M (resp., X) by Γ. The following diagram commutes:*

$$\begin{tikzcd}
HC^{per}_0(\widehat{\mathcal{W}}_{Q_X}(0)) \arrow[r, "\sigma_*^{\widehat{\mathcal{W}}(0)}"] \arrow[d, "\iota_*^{\hbar^{-1},\widehat{\mathcal{W}}}"'] & H^{-\bullet}(\Omega^\bullet_{IQ_X}((u)), d) \arrow[d, "\wedge\frac{1}{m}\mathrm{eu}_{\mathrm{Q}}(\mathrm{N})\wedge\pi^{-1}Td_{IQ_M}"] \\
HC^{per}_0(\widehat{\mathcal{W}}_{Q_X}) \arrow[r, "\mu^{\widehat{\mathcal{W}}}"] & H^{-\bullet}(\Omega^\bullet_{IQ_X}((\hbar))((u)), d)
\end{tikzcd}$$

Proof. Let $i^{IQ} : IQ_X \to Q_X$ be the natural forgetful map. The key observation is that the quasi-isomorphisms $\sigma_*^{\widehat{\mathcal{W}}(0)}$ and $\mu^{\widehat{\mathcal{W}}}$ constructed in [PPT2, Theorem 5.13] are morphisms of $i_*^{IQ_X}\underline{\mathbb{C}}_{IQ_X}((u))$-modules on Q_X (where $\underline{\mathbb{C}}_{IQ_X}((u))$ denotes the (locally) constant sheaf whose space of sections over any connected open subset of IQ_X is $\mathbb{C}((u))$).

For an element x of $H^{-\bullet}(IQ_X;\mathcal{HC}^{per}(\widehat{\mathcal{W}}_{Q_X}(0)))$, $\sigma_*^{\widehat{\mathcal{W}}(0)}(x)$ is an element of $H^{-\bullet}(IQ_X;\underline{\mathbb{C}}((u)))$. Let $\mathbf{1}$ denote the trivial $\widehat{\mathcal{W}}_{Q_X}(0)$-module. Then, we notice that $\sigma_*^{\widehat{\mathcal{W}}(0)}$ maps $\sigma_*^{\widehat{\mathcal{W}}(0)}(x) \cup \mathrm{ch}^{\widehat{\mathcal{W}}(0)}(\mathbf{1})$, as an element in $HC^{per}_0(\widehat{\mathcal{W}}_{Q_X}(0))$, also to $\sigma_*^{\widehat{\mathcal{W}}(0)}(x)$ in $H^{-\bullet}(IQ_X;\underline{\mathbb{C}}((u)))$, where $\cup$ is the cup product

$$\cup : H^{-\bullet}(IQ_X;\underline{\mathbb{C}}((u))) \otimes H^{-\bullet}(Q_X;\mathcal{HC}^{per}(\widehat{\mathcal{W}}_{Q_X}(0))) \to H^{-\bullet}(Q_X;\mathcal{HC}^{per}(\widehat{\mathcal{W}}_{Q_X}(0)))\,.$$

As $\sigma_*^{\widehat{\mathcal{W}}(0)}$ is a quasi-isomorphism, we conclude that $x = [\sigma_*^{\widehat{\mathcal{W}}(0)}(x)] \cup [\mathrm{ch}^{\widehat{\mathcal{W}}(0)}(\mathbf{1})]$ is in $H^{-\bullet}(Q_X;\mathcal{HC}^{per}(\widehat{\mathcal{W}}_{Q_X}(0)))$. Since $\mu^{\widehat{\mathcal{W}}}$ is a $H^{-\bullet}(IQ_X, \underline{\mathbb{C}}((u)))$-module map, we have

$$\mu^{\widehat{\mathcal{W}}}(\iota_*^{\hbar^{-1},\widehat{\mathcal{W}}}(x)) = \sigma_*^{\widehat{\mathcal{W}}(0)}(x) \cup \mu^{\widehat{\mathcal{W}}}(\iota_*^{\hbar^{-1},\widehat{\mathcal{W}}}(\mathrm{ch}^{\widehat{\mathcal{W}}(0)}(\mathbf{1})))\,.$$

This reduces the proof to computing $\mu^{\widehat{\mathcal{W}}}(\iota_*^{\hbar^{-1},\widehat{\mathcal{W}}}(\mathrm{ch}^{\widehat{\mathcal{W}}(0)}(\mathbf{1})))$. This computation is done by the same proof as that of [PPT2, Theorem 5.13], but in the holomorphic setting. As is computed in [BNT, 4.5.1], the characteristic class of the quantization $\widehat{\mathcal{W}}_X(0)$ is equal to $\frac{1}{\hbar}\omega + \frac{1}{2}\pi_M^* c_1(TX)$ with ω the symplectic form on T^*M. Substituting this characteristic class into [PPT2, Theorem 5.13] yields the desired identity. $\square$

Proof of Theorem 4.2: The Euler class is the top degree component of the Chern character, which is computed by the following steps.

$$\begin{aligned}
\mathrm{ch}_Q(\mathcal{M}) &= \mu^{\widehat{\mathcal{E}}} \circ \mathrm{ch}^{\widehat{\mathcal{E}}}(\widehat{\mathcal{M}}) && \text{Definition 3.13}\\
&= \mu^{\widehat{\mathcal{W}}} \circ i_*^{\widehat{\mathcal{E}}} \mathrm{ch}^{\widehat{\mathcal{E}}}(\mathcal{M}) && \text{Proposition 4.1}\\
&= \mu^{\widehat{\mathcal{W}}} \circ i_*^{\mathcal{R}\widehat{\mathcal{E}}[\hbar^{-1}]} \circ i_*^{\hbar^{-1},\widehat{\mathcal{E}}} \mathrm{ch}^{\widehat{\mathcal{E}}}(\mathcal{M}) && \text{right front triangle of Proposition 4.9}\\
&= \mu^{\widehat{\mathcal{W}}} \circ i_*^{\mathcal{R}\widehat{\mathcal{E}}[\hbar^{-1}]} \circ \iota_*^{\hbar^{-1},\mathcal{R}\widehat{\mathcal{E}}} \mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X}) && \text{Proposition 4.6}\\
&= \mu^{\widehat{\mathcal{W}}} \circ \iota_*^{\hbar^{-1},\widehat{\mathcal{W}}} \circ i_*^{\mathcal{R}\widehat{\mathcal{E}}} \circ \mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X}) && \text{right front square of Proposition 4.9}\\
&= \frac{1}{m}\sigma_*^{\widehat{\mathcal{W}}(0)} \circ i_*^{\mathcal{R}\widehat{\mathcal{E}}} \circ \mathrm{ch}^{\mathcal{R}\widehat{\mathcal{E}}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X}) \wedge \mathrm{eu}_Q(N) \wedge \pi^{-1} Td_{IQ_M} && \text{Theorem 4.10}\\
&= \frac{1}{m}\mathrm{ch}^{\mathcal{O}}(\mathcal{R}\widehat{\mathcal{M}}_{Q_X} \otimes_{\widehat{\mathcal{E}}_{Q_X}} \mathcal{O}_{Q_X}) \wedge \mathrm{eu}_Q(N) \wedge \pi^{-1} Td_{IQ_M} && \text{Proposition 4.7}\\
&= \frac{1}{m}\mathrm{ch}^{\mathcal{O}}(\mathrm{Gr}\widehat{\mathcal{M}}_{Q_X} \otimes_{\mathrm{Gr}\widehat{\mathcal{E}}_{Q_X}} \mathcal{O}_{Q_X}) \wedge \mathrm{eu}_Q(N) \wedge \pi^{-1} Td_{IQ_M} && \text{Eq. (8)}\\
&= \frac{1}{m}\mathrm{ch}_Q(\sigma_{\mathrm{char}(\mathcal{M})}(\mathcal{M})) \wedge \mathrm{eu}_Q(N) \wedge \pi^{-1} Td_{IQ_M}. && \square
\end{aligned}$$

References

[BNT] P. Bressler, R. Nest, and B. Tsygan, *Riemann-Roch theorems via deformation quantization*, I, II. Adv. Math. **167** (2002), no. 1, 1-25, 26-73.

[DE] V. Dolgushev and P. Etingof, *Hochschild cohomology of quantized symplectic orbifolds and the Chen-Ruan cohomology*, Int. Math. Res. Not. 2005, no. 27, 1657-1688.

[EF] M. Engeli and G. Felder, *A Riemann-Roch-Hirzebruch formula for traces of differential operators*, Ann. Sci. Éc. Norm. Supér. (4) **41**(2008), No. 4, 621-653.

[F] B. V. Fedosov, *On G-trace and G-index in deformation quantization*, Letters in Math. Phys. **52** (2000), 29-49.

[FFS] B. Feigin, G. Felder and B. Shoikhet, *Hochschild cohomology of the Weyl algebra and traces in deformation quantization*, Duke Math. J. **127** (2005), No. 3, 487-517.

[FT] G. Felder, X. Tang, *Equivariant Lefschetz number of differential operators*, Math. Z., **266** (2010), no. 2, 451-470.

[G] S. Guillermou, *Lefschetz class of elliptic pairs*, Duke Math. J. **85** (1996), no. 2, 273-314.

[K] M. Kashiwara, *D-modules and Microlocal Calculus*, Translations of Mathematical Monographs, **217** American Math. Soc. (2003).

[KS] M. Kashiwara and P. Schapira, *Deformation quantization modules*, arXiv:1003.3304, to appear in Astésrique.

[NPPT] N. Neumaier, M. Pflaum, H. Posthuma and X. Tang, *Homology of formal deformations of proper étale groupoids*, J. Reine Angew. Math. **593**, 117-168 (2006).

[PPT1] M. Pflaum, H. Posthuma and X. Tang, *An algebraic index theorem for orbifolds*, Adv. Math. **210** (2007), No. 1, 83-121.

[PPT2] M. Pflaum, H. Posthuma and X. Tang, *Cyclic cocycles in deformation quantization and higher index theorems*, Adv. Math. **223** (2010), No. 6, 1958-2021.

[PS] P. Polesello and P. Schapira, *Stacks of quantization-deformation modules over complex symplectic manifolds*, Int. Math. Res. Notices **49**, 2637-2664 (2004).

[R] A. Ramadoss, *Some notes on the Feigin-Losev-Shoikhet integral conjecture*, J. Noncommut. Geom. **2** (2008), 405-448.

[SS1] P. Schapira and J. Schneider, *Elliptic pairs. I. Relative finiteness and duality. Index theorem for elliptic pairs.*, Astérisque No. **224** (1994), 5-60.

[SS2] P. Schapira, J. Schneiders, *Elliptic pairs. II. Euler class and relative index theorem. Index theorem for elliptic pairs*, Astérisque No. **224** (1994), 61-98.

Departement Mathematik, ETH Zürich, Rämistrasse 101, 8092 Zürich
E-mail address: ajay.ramadoss@math.ethz.ch

Department of Mathematics, Washington University, St. Louis, MO 63130, USA
E-mail address: xtang@math.wustl.edu

Department of Mathematics, Ohio State University, Columbus, OH 43210, USA
E-mail address: hhtseng@math.ohio-state.edu